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EFFECTIVE STIFFNESSES OF TRANSVERSELY NON-HOMOGENEOUS PLATES WITH UNIDIRECTIONAL PERIODIC STRUCTURE

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Abstract—This paper presents a derivation and analysis of closed formulae for the membrane, bending and reciprocal effective stiffnesses of elastic orthotropic and transversely asymmetric plates with microstructures periodic in one direction. The derivation is based on the concept of imposing Hencky-type displacement constraints on the solutions to the Caillerie–Kohn–Vogelius local homogenization problems. Reduction of the dimension of the problems by one makes it possible to solve the new approximate local problems exactly, thus enabling one to find the relevant formulae in closed form, ready for engineering applications as well as for optimization and sensitivity studies. This paper generalizes the previous results of the author [*Int. J. Solids Structures* **29**, 309–326 (1992)], concerning the bending of transversely symmetric periodic plates.

1. INTRODUCTION

The problem of evaluating the effective stiffnesses of thin plates of regular repeated structure and constant thickness has been solved by Caillerie (1984, model $e \approx \varepsilon$). A solution to the similar problem concerning thin homogeneous plates of periodically varying thickness can be found in Kohn and Vogelius (1984, 1985, 1986a, model $a = 1$). The other models discussed in the above papers refer to very slender or very flat shapes of the cell of periodicity and hence cannot be viewed as solutions to the original, generally posed problem.

Effective stiffnesses of plates depend upon the spatial characteristics of the periodicity cells. A mathematical consequence is that the local homogenization problems derived by Caillerie, Kohn and Vogelius are posed on the spatial, rescaled periodicity cell. In view of the usual irregularities of shape or physical properties of this cell, caused by varying thickness, the presence of a reinforcement, etc., as well as possible discrepancies between the material characteristics of the matrix and the inclusions, the exact solutions to the local homogenization problems cannot be found and even finite element approximations can only then be satisfactory if appropriate mesh refinements are adopted [cf. Guedes and Kikuchi (1990), where similar problems, circumventing the Caillerie–Kohn–Vogelius algorithm, have been considered]. The difficulties arising in adhering to the Caillerie–Kohn–Vogelius algorithm justify attempts to form approximate methods based on this algorithm.

In the case when the periodicity cells can be viewed as thick plates, solutions to Caillerie–Kohn–Vogelius local problems may be predicted similarly as in the theories of plates with transverse shear deformation, especially by the Hencky (1947) plate model [cf. Lewiński (1991a, Part III, Sect. 5) and Telega (1992)]. The dimension of the local problems constructed in this manner is then reduced by one: the unknown new auxiliary fields, representing the average rotations of cross-sections and the deflection of the periodicity cell, are defined on the plane rectangular reference domain. As has been shown in Lewiński (1991a, Part III; 1992, 1993), the stiffnesses found in this manner are sensitive to the transverse proportions of the original three-dimensional periodicity cell. On the other hand, the formulae obtained by imposing Kirchhoff constraints upon the same local problems turn out to coincide with formulae found originally by Duvaut and Metellus (1976), being insensitive to these proportions, which essentially limits their range of applicability to the case of periodicity cells which are themselves extremely flat plates.

The significance of formulae based on the aforementioned method of averaging follows from its efficiency in application to an important class of problems of averaging stiffnesses of plates periodic in one direction. The local homogenization problems can then be reduced to solving the sets of ordinary differential equations with variable coefficients. In many cases these equations can be solved analytically and consequently the formulae for effective stiffnesses assume closed forms. The significance of such formulae surpasses the usual needs of engineering computations. These formulae are crucial in determining the effective stiffnesses of the so-called ribbed plates of finite rank—a plate theory analogue of the laminates of finite rank. Introduction of such composite plates into the formulation of the compliance minimization problem regularizes the original problem, i.e. it makes the problem uniquely solvable. This method of regularization originated in the papers of Cheng (1981) and Cheng and Olhoff (1981) and was then developed by Bendsøe (1987), Bendsøe *et al.* (1993), Kohn and Strang (1986), Kohn and Vogelius (1986b), Lur'e and Cherkaev (1986), Rozvany *et al.* (1987), Bonnetier and Vogelius (1987), Lipton (1994), Allaire and Kohn (1993, case of plane stress), Soto and Diaz (1992), and Thomsen (1992). Knowledge of the analytical formulae for ribbed plates enables one to subsequently determine the average stiffnesses of composite plates obtained by subsequent introduction of ribs in alternate directions. Thus, a partly analytical treatment of such a regularized problem is possible. Similarly, a partly analytical sensitive analysis becomes available.

Effective stiffnesses of transversely symmetric moderately thick plates regularly and symmetrically ribbed in one direction have been examined in Lewiński (1992). The aim of the present paper is to derive relevant formulae for transversely asymmetric plates. Use will be made of the Hencky approximation applied to Caillerie–Kohn–Vogelius local homogenization problems. This method, discussed briefly in Lewiński (1991b), will be presented in detail and illustrated with examples of plates used in engineering practice.

A brief outline of the method used and a characterization of results obtained concerning effective stiffnesses is now given. The method consists of several steps that may be described as follows:

(i) The point of departure: a three-dimensional variational formulation of the equilibrium problem of a linearly elastic, transversely unsymmetric (unbalanced) plate with properties Y_α-periodic in the in-plane x_α direction; the plate is composed of a three-dimensional periodicity cell $\mathscr{Y}$ by repetition along the x_α axes.

(ii) Introducing a small parameter ε; all dimensions of the periodicity cell of the plate are now multiplied by this parameter. The problem becomes εY_α-periodic.

(iii) expanding the solution of this ε-dependent problem by introducing rapid variables $y_i = x_i/\varepsilon$ and assuming a specific two-scale representation.

(iv) Performing the small parameter analysis and finding equations governing the first terms of the two-scale representations. These equations, forming a set of Kirchhoff-type plate equations, refer to a hypothetical thin transversely asymmetric plate with anisotropic stiffness tensors determined by formulae involving some special auxiliary functions. These functions are determined implicitly as solutions to the three-dimensional problems (called local problems of Caillerie–Kohn–Vogelius) posed on the three-dimensional rescaled cell of periodicity $\mathscr{Y}$. In the case of one-directional periodicity, this cell is two-dimensional. It is rescaled or ε-independent.

(v) Assuming that $\mathscr{Y}$ has the shape of a plate, one may approximate the local problems by imposing a linear distribution of the unknown auxiliary functions in the transverse dimension and unknown distribution in the in-plane directions. This can be viewed as Hencky's two-dimensional modelling. In the case of one-directional periodicity this modelling reduces the Caillerie–Kohn–Vogelius local problems to two problems involving unknown functions of one variable $y_1 : (T_\gamma^{(\alpha\beta)}, Z_\gamma^{(\alpha\beta)}, X^{(\alpha\beta)})$ and $(U_\gamma^{(\alpha\beta)}, \Phi_\gamma^{(\alpha\beta)}, W^{(\alpha\beta)})$, with α, β and γ assuming values 1 or 2, respectively. The mathematical structure of these problems is similar to that of the coupled problem of stretching, bending and transverse shearing of a non-homogeneous Timoshenko beam.

(vi) The unknown functions indexed by $(\alpha,\beta) = (1,1)$ or $(2,2)$ enter into the definitions of effective stiffnesses of indices $\alpha\alpha\beta\beta$. These formulae can be found analytically irrespective of the variation of stiffnesses along y_1.

(vii) the unknown functions indexed by $(\alpha,\beta) = (1,2)$ or $(2,1)$ are involved in the definitions of effective stiffnesses of indices 1212, or the in-plane shearing, reciprocal and torsional effective stiffnesses. These functions cannot generally be found analytically. In the case of a piecewise constant variation of stiffnesses along y_1, e.g. resulting from rapid changes of the plate thickness, the local problems can be solved analytically and the effective stiffnesses cast in closed form expressions. The formulae thus derived turn out to be sensitive to the transverse shape of $\mathscr{Y}$, in contrast to the formulae found previously by Duvaut and Metellus (1976) and Bourgeat and Tapiéro (1983).

(viii) One can prove that the tensors of effective stiffnesses found by the method outlined above satisfy usual symmetry and positive definiteness conditions, thus enabling the homogenized problem to be correctly posed. In the transverse symmetry (or balanced) case, the formulae coincide with those reported previously in Lewiński (1992).

The usual summation convention for the indices at different levels is adopted. Small Greek letters (except for ε) run over 1,2; the Latin ones take values of 1,2,3. Moreover, the following abbreviations for the partial derivatives are used: $\partial/\partial x_j = {}_{,j}$, $\partial/\partial y_j = {}_{|j}$.

2. AVERAGING OF STIFFNESSES BY THE CAILLERIE–KOHN–VOGELIUS METHOD

Let us consider a plate occupying the domain

$$B_\varepsilon = \left\{ \mathbf{x} = (x, x_3) \,|\, x = (x_\alpha) \in \Omega, \quad c^-\left(\frac{x}{\varepsilon}\right) < x_3 < c^+\left(\frac{x}{\varepsilon}\right) \right\},$$

where Ω is a reference plane, x_i are Cartesian coordinates, and ε is a small parameter. The functions $c^\pm(\cdot)$ are Y-periodic, $Y = (0, Y_1) \times (0, Y_2)$. Elastic moduli $C^{ijkl}(x, x_3)$ are εY-periodic with respect to x. Thus, there exist functions $C^{ijkl}(\cdot,\cdot)$ Y-periodic with respect to the first variable, such that

$$C^{ijkl}(x, x_3) = C^{ijkl}\left(\frac{x}{\varepsilon}, \frac{x_3}{\varepsilon}\right). \tag{1}$$

We only consider the case when the $x_3 = \text{const}$ planes are planes of material symmetry, i.e.

$$C^{3\alpha\beta\lambda} = C^{333\alpha} = 0. \tag{2}$$

Let us define

$$\tilde{C}^{\alpha\beta\lambda\mu} = C^{\alpha\beta\lambda\mu} - c^{\alpha\beta} C^{33\lambda\mu}, \quad c^{\alpha\beta} = C^{33\alpha\beta}/C^{3333}. \tag{3}$$

Tensor $\tilde{\mathbf{C}}$ refers to the plane stress approximation. Plate B_ε is composed of the identical segments $\varepsilon\mathscr{Y}$, the rescaled cell $\mathscr{Y}$ being described by

$$\mathscr{Y} = \{ \mathbf{y} | \mathbf{y} = (y, y_3), \quad y = (y_1, y_2) \in Y, \quad c^-(y) < y_3 < c^+(y) \}. \tag{4}$$

Here $y_i = x_i/\varepsilon$. We note that fields C^{ijkl} are defined on $\mathscr{Y}$.

The asymptotic method discussed in Caillerie (1984), Kohn and Vogelius (1984) [cf. also Kalamkarov *et al.* (1987) and Lewiński (1991a,b)] makes it possible to decompose the problem originally posed on the B_ε domain into a sequence of two-dimensional problems posed on the plane domain Ω. The effective stiffnesses in these problems are determined by the solutions to the auxiliary three-dimensional local problems posed on $\mathscr{Y}$. We shall recall below the formulations of these problems after Lewiński (1991a,b).

Let us define a function space

$$W(\mathscr{Y}) = \{\mathbf{v} = (v_i) \in [H^1(\mathscr{Y})]^3 \,|\, \mathbf{v}\text{ assumes equal values at opposite lateral edges of }\mathscr{Y}\}$$

and a bilinear form

$$a(\mathbf{u},\mathbf{v}) = \langle C^{ijkl}(\mathbf{y}) u_{i|j} v_{k|l} \rangle, \quad \mathbf{u},\mathbf{v} \in W(\mathscr{Y}), \tag{5}$$

where the brackets $\langle \cdot \rangle$ represent averaging over $\mathscr{Y}$:

$$\langle \cdot \rangle = \frac{1}{\mathrm{vol}(\mathscr{Y})} \int_{\mathscr{Y}} (\cdot)\, \mathrm{d}\mathscr{Y}. \tag{6}$$

Two local problems assume the form

$$\text{find } \Theta^{(\alpha\beta)} \in W(\mathscr{Y}) \text{ such that}$$

$$(P^1_{\mathrm{loc}}) \quad a(\Theta^{(\alpha\beta)}, \mathbf{w}) + \langle C^{ij\alpha\beta} w_{i|j} \rangle = 0 \quad \forall \mathbf{w} \in W(\mathscr{Y});$$

$$\text{find } \Xi^{(\alpha\beta)} \in W(\mathscr{Y}) \text{ such that}$$

$$(P^2_{\mathrm{loc}}) \quad a(\Xi^{(\alpha\beta)}, \mathbf{w}) + \langle \hat{y}_3 C^{ij\alpha\beta} w_{i|j} \rangle = 0 \quad \forall \mathbf{w} \in W(\mathscr{Y}),$$

where $\hat{y}_3 = y_3 - \langle y_3 \rangle$. Solutions to these problems are determined up to additive constants. To make them unique we impose the following "oscillation conditions":

$$\langle \Theta^{(\alpha\beta)} \rangle = 0, \quad \langle \Xi^{(\alpha\beta)} \rangle = 0. \tag{7}$$

The effective stiffnesses, referring to the rescaled plate composed of cells $\mathscr{Y}$, are expressed as follows:

$$\begin{aligned}
A_z^{\alpha\beta\lambda\mu} &= \langle \Sigma^{\lambda\mu}_{1(\alpha\beta)} \rangle, \quad \Sigma^{\lambda\mu}_{1(\alpha\beta)} = C^{\alpha\beta kl} \Theta^{(\lambda\mu)}_{k|l} + C^{\alpha\beta\lambda\mu}, \\
F_z^{\alpha\beta\lambda\mu} &= \langle \hat{y}_3 \Sigma^{\lambda\mu}_{1(\alpha\beta)} \rangle, \\
E_z^{\alpha\beta\lambda\mu} &= \langle \Sigma^{\lambda\mu}_{2(\alpha\beta)} \rangle, \quad \Sigma^{\lambda\mu}_{2(\alpha\beta)} = C^{\alpha\beta kl} \Xi^{(\lambda\mu)}_{k|l} + \hat{y}_3 C^{\alpha\beta\lambda\mu}, \\
D_z^{\alpha\beta\lambda\mu} &= \langle \hat{y}_3 \Sigma^{\lambda\mu}_{2(\alpha\beta)} \rangle.
\end{aligned} \tag{8}$$

The effective stiffnesses of plate B_ε are given by

$$\begin{aligned}
A^{\alpha\beta\lambda\mu}_{\mathrm{hom}} &= \varepsilon t A_z^{\alpha\beta\lambda\mu}, \quad F^{\alpha\beta\lambda\mu}_{\mathrm{hom}} = \varepsilon^2 t F_z^{\alpha\beta\lambda\mu}, \\
E^{\alpha\beta\lambda\mu}_{\mathrm{hom}} &= \varepsilon^2 t E_z^{\alpha\beta\lambda\mu}, \quad D^{\alpha\beta\lambda\mu}_{\mathrm{hom}} = \varepsilon^3 t D_z^{\alpha\beta\lambda\mu},
\end{aligned} \tag{9}$$

where $t = \mathrm{vol}(\mathscr{Y})/\mathrm{area}(Y)$. Let $N^{\alpha\beta}$, $M^{\alpha\beta}$ represent the effective membrane forces and moments referred to the $\hat{y}_3 = 0$ plane, while $\gamma_{\alpha\beta}$, $\kappa_{\alpha\beta}$ represent the in-plane and anti-plane deformation measures referred to the same plane. The constitutive relationships assume the form

$$N^{\alpha\beta} = A_{\text{hom}}^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + E_{\text{hom}}^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu},$$

$$M^{\alpha\beta} = F_{\text{hom}}^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + D_{\text{hom}}^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}. \tag{10}$$

One can prove that the stiffness tensors involved in eqns (10) satisfy the following symmetry conditions:

$$A_{\approx}^{\alpha\beta\lambda\mu} = A_{\approx}^{\lambda\mu\alpha\beta}, \quad E_{\approx}^{\alpha\beta\lambda\mu} = F_{\approx}^{\lambda\mu\alpha\beta}, \quad D_{\approx}^{\alpha\beta\lambda\mu} = D_{\approx}^{\lambda\mu\alpha\beta},$$

$$K_{\approx}^{\alpha\beta\lambda\mu} = K_{\approx}^{\beta\alpha\lambda\mu} = K_{\approx}^{\beta\alpha\mu\lambda}, \quad K = A, E, F, D \qquad \text{(11a–e)}$$

and that a constant $c > 0$ exists such that for all $\gamma = (\gamma_{\alpha\beta})$, $\kappa = (\kappa_{\alpha\beta}) \in M_s^2$ (M_s^2 represents the space of 2×2 symmetric matrices), the following estimate holds:

$$W(\gamma, \kappa) = A_{\approx}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + E_{\approx}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\kappa_{\lambda\mu} + F_{\approx}^{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\gamma_{\lambda\mu} + D_{\approx}^{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu} \geqslant c\sum_{\alpha,\beta}(\gamma_{\alpha\beta}\gamma_{\alpha\beta} + \kappa_{\alpha\beta}\kappa_{\alpha\beta}). \tag{12}$$

3. AN APPROXIMATE ALGORITHM

3.1. *Simplification of the local analysis*

The simplified analysis presented below refers to the case when the cell $\mathscr{Y}$ can be viewed as a moderately thick plate and moduli $c^{\alpha\beta}$ do not depend upon y_1, y_2, i.e.

$$c^{\alpha\beta} = c^{\alpha\beta}(y_3). \tag{13}$$

A more stringent restriction, $c^{\alpha\beta} = \text{const}$, has been introduced in Lewiński (1991, Part III). The approximation used here will not result in violation of the symmetry and positive definiteness conditions (12) and (13).

The solutions to the $(P_{\text{loc}}^{\alpha})$ problems can be decomposed as follows:

$$\Theta^{(\alpha\beta)} = \bar{\Theta}^{(\alpha\beta)} + \tilde{\Theta}^{(\alpha\beta)}, \quad \Xi^{(\alpha\beta)} = \bar{\Xi}^{(\alpha\beta)} + \tilde{\Xi}^{(\alpha\beta)}, \tag{14}$$

with

$$\tilde{\Theta}^{(\alpha\beta)} = (0, 0, -\textstyle\int c^{\alpha\beta}\,\mathrm{d}y_3), \quad \tilde{\Xi}^{(\alpha\beta)} = (0, 0, -\textstyle\int \hat{y}_3 c^{\alpha\beta}\,\mathrm{d}y_3), \tag{15}$$

fields $\bar{\Theta}^{(\alpha\beta)}$, $\bar{\Xi}^{(\alpha\beta)}$ being solutions to the modified local problems:

$(\bar{P}_{\text{loc}}^{1})$ find $\bar{\Theta}^{(\alpha\beta)} \in W(\mathscr{Y})$ such that

$$a(\bar{\Theta}^{(\alpha\beta)}, \mathbf{w}) + \langle \tilde{C}^{\gamma\delta\alpha\beta} w_{\gamma|\delta} \rangle = 0 \quad \forall \mathbf{w} \in W(\mathscr{Y});$$

$(\bar{P}_{\text{loc}}^{2})$ find $\bar{\Xi}^{(\alpha\beta)} \in W(\mathscr{Y})$ such that

$$a(\bar{\Xi}^{(\alpha\beta)}, \mathbf{w}) + \langle \hat{y}_3 \tilde{C}^{\gamma\delta\alpha\beta} w_{\gamma|\delta} \rangle = 0 \quad \forall \mathbf{w} \in W(\mathscr{Y}).$$

The decomposition (14) follows from the following identities:

$$C^{\alpha\beta ij} w_{i|j} = \tilde{C}^{\alpha\beta\gamma\delta} w_{\gamma|\delta} + c^{\alpha\beta} C^{33ij} w_{i|j}, \quad \tilde{\Theta}_{3|3}^{(\alpha\beta)} = -c^{\alpha\beta}(y_3), \quad \tilde{\Xi}_{3|3}^{(\alpha\beta)} = -\hat{y}_3 c^{\alpha\beta}(y_3). \tag{16}$$

Solutions to the $(\bar{P}_{\text{loc}}^{\alpha})$ problems are determined up to additive constants.

Assume that the cell of periodicity $\mathscr{Y}$ has properties of a plate of moderate thickness. Then problems $(\bar{P}^{\alpha}_{\text{loc}})$ can be viewed as plate-type problems. The loadings involved in these problems can be treated as initial stresses acting in the y_3 = const plane and consequently the fields $\bar{\Theta}^{(\alpha\beta)}$, $\bar{\Xi}^{(\alpha\beta)}$ will represent displacement fields. Upon these fields the Hencky-type constraints can be imposed:

$$\bar{\Theta}^{(\alpha\beta)}_{\lambda}(\mathbf{y}) = T^{(\alpha\beta)}_{\lambda}(y)+\hat{y}_3 Z^{(\alpha\beta)}_{\lambda}(y), \quad \bar{\Theta}^{(\alpha\beta)}_{3}(\mathbf{y}) = X^{(\alpha\beta)}(y),$$
$$\bar{\Xi}^{(\alpha\beta)}_{\lambda}(\mathbf{y}) = U^{(\alpha\beta)}_{\lambda}(y)+\hat{y}_3 \Phi^{(\alpha\beta)}_{\lambda}(y), \quad \bar{\Xi}^{(\alpha\beta)}_{3}(\mathbf{y}) = W^{(\alpha\beta)}(y). \tag{17}$$

The fields $T^{(\alpha\beta)}_{\lambda}, Z^{(\alpha\beta)}_{\lambda}, X^{(\alpha\beta)}$; $U^{(\alpha\beta)}_{\lambda}, \Phi^{(\alpha\beta)}_{\lambda}, W^{(\alpha\beta)}$ are new unknown fields of class $H^1_{\text{per}}(Y)$ [functions from $H^1(Y)$ assuming equal traces at opposite sides of Y], representing rotations of cross-sections or deflections of $\mathscr{Y}$. Similar constraints are imposed upon the trial fields:

$$w_{\lambda}(\mathbf{y}) = u_{\lambda}(y)+\hat{y}_3\varphi_{\lambda}(y), \quad w_3(\mathbf{y}) = w(y), \tag{18}$$

where $u_{\lambda}, \varphi_{\lambda}, w \in H^1_{\text{per}}(Y)$.

In Hencky's theory of plates, one also introduces the assumption of negligibility of transverse normal stresses, i.e. $\sigma^{33} \approx 0$, which modifies the constitutive relationships. Similarly, here it will be assumed that the fields

$$\bar{\Sigma}^{33}_{1(\lambda\mu)} = C^{33\gamma\delta}\bar{\Theta}^{(\lambda\mu)}_{\gamma|\delta}+C^{3333}\bar{\Theta}^{(\lambda\mu)}_{3|3}, \quad \bar{\Sigma}^{33}_{2(\lambda\mu)} = C^{33\gamma\delta}\bar{\Xi}^{(\lambda\mu)}_{\gamma|\delta}+C^{3333}\bar{\Xi}^{(\lambda\mu)}_{3|3} \tag{19}$$

are negligible, which makes it possible to express the quantities $\bar{\Sigma}^{(\alpha\beta)}_{\gamma(\lambda\mu)}$ by the following formulae:

$$\bar{\Sigma}^{\alpha\beta}_{1(\lambda\mu)} = C^{\alpha\beta kl}\bar{\Theta}^{(\lambda\mu)}_{k|l} \approx \tilde{C}^{\alpha\beta\gamma\delta}\bar{\Theta}^{(\lambda\mu)}_{\gamma|\delta}, \quad \bar{\Sigma}^{\alpha\beta}_{2(\lambda\mu)} = C^{\alpha\beta kl}\bar{\Xi}^{(\lambda\mu)}_{k|l} \approx \tilde{C}^{\alpha\beta\gamma\delta}\bar{\Xi}^{(\lambda\mu)}_{\gamma|\delta}. \tag{20}$$

The kinematical as well as stress assumptions adopted make it possible to reduce the transverse dimension in $(\bar{P}^{\alpha}_{\text{loc}})$. Prior to formulating the problems reduced in this manner, let us define the stiffnesses

$$\begin{bmatrix} A^{\alpha\beta\lambda\mu} \\ E^{\alpha\beta\lambda\mu} \\ D^{\alpha\beta\lambda\mu} \end{bmatrix} = \int_{c^-(y)}^{c^+(y)} \tilde{C}^{\alpha\beta\lambda\mu}(\mathbf{y}) \begin{bmatrix} 1 \\ \hat{y}_3 \\ (\hat{y}_3)^2 \end{bmatrix} \mathrm{d}y_3, \quad H^{\lambda\mu} = \int_{c^-(y)}^{c^+(y)} C^{\lambda 3\mu 3}(\mathbf{y})\,\mathrm{d}y_3 \tag{21a,b}$$

and the bilinear forms

$$A(\mathbf{u},\mathbf{v}) = \{A^{\lambda\mu\gamma\delta}u_{\gamma|\delta}v_{\lambda|\mu}\}, \quad E(\mathbf{u},\mathbf{v}) = \{E^{\lambda\mu\gamma\delta}u_{\gamma|\delta}v_{\lambda|\mu}\},$$
$$D(\mathbf{u},\mathbf{v}) = \{D^{\lambda\mu\gamma\delta}u_{\gamma|\delta}v_{\lambda|\mu}\}, \quad G(\mathbf{u},\mathbf{v}) = \{H^{\alpha\beta}u_{\beta}v_{|\alpha}\},$$
$$H(u,v) = \{H^{\lambda\mu}u_{|\lambda}v_{|\mu}\}, \quad J(\mathbf{u},\mathbf{v}) = \{H^{\lambda\mu}v_{\lambda}u_{\mu}\}, \tag{22}$$

where $\mathbf{u},\mathbf{v} \in [H^1_{\text{per}}(Y)]^2$, $u,v \in H^1_{\text{per}}(Y)$. The braces mean the averaging over Y:

$$\{\cdot\} = \frac{1}{Y_1 Y_2}\int_{Y}(\cdot)\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

Let us introduce the fields that will have the meaning of membrane forces and moments:

$$N^{\lambda\mu}_{1(\alpha\beta)} = A^{\lambda\mu\gamma\delta}T^{(\alpha\beta)}_{\gamma|\delta}+E^{\lambda\mu\gamma\delta}Z^{(\alpha\beta)}_{\gamma|\delta}+A^{\lambda\mu\alpha\beta},$$
$$M^{\lambda\mu}_{1(\alpha\beta)} = E^{\lambda\mu\gamma\delta}T^{(\alpha\beta)}_{\gamma|\delta}+D^{\lambda\mu\gamma\delta}Z^{(\alpha\beta)}_{\gamma|\delta}+E^{\lambda\mu\alpha\beta},$$

$$N^{\lambda\mu}_{2(\alpha\beta)} = A^{\lambda\mu\gamma\delta}U^{(\alpha\beta)}_{\gamma|\delta} + E^{\lambda\mu\gamma\delta}\Phi^{(\alpha\beta)}_{\gamma|\delta} + E^{\lambda\mu\alpha\beta},$$

$$M^{\lambda\mu}_{2(\alpha\beta)} = E^{\lambda\mu\gamma\delta}U^{(\alpha\beta)}_{\gamma|\delta} + D^{\lambda\mu\gamma\delta}\Phi^{(\alpha\beta)}_{\gamma|\delta} + D^{\lambda\mu\alpha\beta},$$

$$Q^{\lambda}_{1(\alpha\beta)} = H^{\lambda\mu}(X^{(\alpha\beta)}_{|\mu} + Z^{(\alpha\beta)}_{\mu}), \quad Q^{\lambda}_{2(\alpha\beta)} = H^{\lambda\mu}(W^{(\alpha\beta)}_{|\mu} + \Phi^{(\alpha\beta)}_{\mu}). \tag{23a–f}$$

With this notation the variational problems $(\bar{P}^{\sigma}_{loc})$ assume the form

$$\{N^{\lambda\mu}_{\sigma(\alpha\beta)}u_{\lambda|\mu} + M^{\lambda\mu}_{\sigma(\alpha\beta)}\varphi_{\lambda|\mu} + Q^{\lambda}_{\sigma(\alpha\beta)}(w_{|\lambda} + \varphi_{\lambda})\} = 0, \tag{24}$$

similar to the Hencky equation.

Problem $(\bar{P}^{1}_{loc})$ is replaced with the problem:

find $(\mathbf{T}^{(\alpha\beta)}, \mathbf{Z}^{(\alpha\beta)}, X^{(\alpha\beta)}) \in H(Y) = [H^{1}_{per}(Y)]^2 \times [H^{1}_{per}(Y)]^2 \times H^{1}_{per}(Y)$ such that

$$(\hat{P}^{1}_{loc}) \quad A(\mathbf{T}^{(\alpha\beta)}, \mathbf{u}) + E(\mathbf{Z}^{(\alpha\beta)}, \mathbf{u}) + \{A^{\gamma\delta\alpha\beta}u_{\gamma|\delta}\} = 0,$$

$$E(\mathbf{T}^{(\alpha\beta)}, \varphi) + D(\mathbf{Z}^{(\alpha\beta)}, \varphi) + G(\varphi, X^{(\alpha\beta)}) + J(\mathbf{Z}^{(\alpha\beta)}, \varphi) + \{E^{\gamma\delta\alpha\beta}\varphi_{\gamma|\delta}\} = 0,$$

$$H(X^{(\alpha\beta)}, w) + G(\mathbf{Z}^{(\alpha\beta)}, w) = 0$$

for all $(\mathbf{u}, \varphi, w) \in H(Y)$,

while the problem $(\bar{P}^{2}_{loc})$ now takes the following form:

find $(\mathbf{U}^{(\alpha\beta)}, \Phi^{(\alpha\beta)}, W^{(\alpha\beta)}) \in H(Y)$ such that

$$(\hat{P}^{2}_{loc}) \quad A(\mathbf{U}^{(\alpha\beta)}, \mathbf{u}) + E(\Phi^{(\alpha\beta)}, \mathbf{u}) + \{E^{\alpha\beta\gamma\delta}u_{\gamma|\delta}\} = 0,$$

$$E(\mathbf{U}^{(\alpha\beta)}, \varphi) + D(\Phi^{(\alpha\beta)}, \varphi) + G(\varphi, W^{(\alpha\beta)}) + J(\Phi^{(\alpha\beta)}, \varphi) + \{D^{\gamma\delta\alpha\beta}\varphi_{\gamma|\delta}\} = 0,$$

$$H(W^{(\alpha\beta)}, w) + G(\Phi^{(\alpha\beta)}, w) = 0$$

for all $(\mathbf{u}, \varphi, w) \in H(Y)$.

Problems $(\hat{P}^{\alpha}_{loc})$ of Hencky type are well posed [see Lagnese and Lions (1988)]. One can prove that fields $\mathbf{T}^{(\alpha\beta)}$, $\mathbf{U}^{(\alpha\beta)}$, $X^{(\alpha\beta)}$, $\mathbf{W}^{(\alpha\beta)}$ are determined up to additive constants, while fields $\mathbf{Z}^{(\alpha\beta)}$, $\Phi^{(\alpha\beta)}$ are determined uniquely.

3.2. *Approximate formulae for effective stiffnesses*

Let us express the effective stiffnesses (8) with the help of solutions of problems $(\hat{P}^{\alpha}_{loc})$. Bearing in mind decompositions (14), one obtains

$$\Sigma^{\lambda\mu}_{1(\alpha\beta)} = C^{\alpha\beta kl}\bar{\Theta}^{(\lambda\mu)}_{k|l} + C^{\alpha\beta kl}\tilde{\Theta}^{(\lambda\mu)}_{k|l} + C^{\alpha\beta\lambda\mu},$$

$$\Sigma^{\lambda\mu}_{2(\alpha\beta)} = C^{\alpha\beta kl}\bar{\Xi}^{(\lambda\mu)}_{k|l} + C^{\alpha\beta kl}\tilde{\Xi}^{(\lambda\mu)}_{k|l} + \hat{y}_3 C^{\alpha\beta\lambda\mu}. \tag{25}$$

Taking into account the approximations (20) and the definition (3) of the tensor $\tilde{\mathbf{C}}$, one finds

$$\Sigma^{\lambda\mu}_{1(\alpha\beta)} = \tilde{C}^{\alpha\beta\gamma\delta}\bar{\Theta}^{(\lambda\mu)}_{\gamma|\delta} + \tilde{C}^{\alpha\beta\lambda\mu}, \quad \Sigma^{\lambda\mu}_{2(\alpha\beta)} = \tilde{C}^{\alpha\beta\gamma\delta}\bar{\Xi}^{(\lambda\mu)}_{\gamma|\delta} + \hat{y}_3\tilde{C}^{\alpha\beta\lambda\mu}. \tag{26}$$

On substitution of the expressions given above into the definition (8), taking into account the hypotheses (17) and performing integration over y_3, one finds the following approximate expressions for the effective stiffness tensors:

$$\tilde{A}^{\alpha\beta\lambda\mu}_{:} = t^{-1}\{N^{\lambda\mu}_{1(\alpha\beta)}\},\quad \tilde{F}^{\alpha\beta\lambda\mu}_{:} = t^{-1}\{M^{\lambda\mu}_{1(\alpha\beta)}\},\quad \tilde{E}^{\alpha\beta\lambda\mu}_{:} = t^{-1}\{N^{\lambda\mu}_{2(\alpha\beta)}\},\quad \tilde{D}^{\alpha\beta\lambda\mu}_{:} = t^{-1}\{M^{\lambda\mu}_{2(\alpha\beta)}\}. \tag{27a–d}$$

The stiffnesses thus defined should possess properties of symmetry (11) and positive definiteness (12). We shall prove that this is the case. By forming appropriate identities following from the variational equations of the $(\hat{P}^{\alpha}_{\mathrm{loc}})$ problems and linking them with formulae (27), one arrives at the following new representations for the effective stiffnesses:

$$\begin{aligned}
t\tilde{A}^{\alpha\beta\lambda\mu}_{:} &= \{A^{\alpha\beta\lambda\mu}\} - A(\mathbf{T}^{(\lambda\mu)},\mathbf{T}^{(\alpha\beta)}) - [E(\mathbf{Z}^{(\lambda\mu)},\mathbf{T}^{(\alpha\beta)}) + E(\mathbf{T}^{(\lambda\mu)},\mathbf{Z}^{(\alpha\beta)})] \\
&\quad - D(\mathbf{Z}^{(\lambda\mu)},\mathbf{Z}^{(\alpha\beta)}) - J(\mathbf{Z}^{(\lambda\mu)},\mathbf{Z}^{(\alpha\beta)} + H(X^{(\alpha\beta)},X^{(\lambda\mu)}) \\
t\tilde{F}^{\lambda\mu\alpha\beta}_{:} &= \{E^{\alpha\beta\lambda\mu}\} - A(\mathbf{U}^{(\alpha\beta)},\mathbf{T}^{(\lambda\mu)}) - [E(\Phi^{(\alpha\beta)},\mathbf{T}^{(\lambda\mu)}) + E(\mathbf{U}^{(\alpha\beta)},\mathbf{Z}^{(\lambda\mu)})] \\
&\quad - D(\Phi^{(\alpha\beta)},\mathbf{Z}^{(\lambda\mu)}) - J(\Phi^{(\alpha\beta)},\mathbf{Z}^{(\lambda\mu)}) + H(X^{(\alpha\beta)},W^{(\lambda\mu)}) \\
t\tilde{E}^{\lambda\mu\alpha\beta}_{:} &= \{E^{\alpha\beta\lambda\mu}\} - A(\mathbf{T}^{(\alpha\beta)},\mathbf{U}^{(\lambda\mu)}) - [E(\mathbf{T}^{(\alpha\beta)},\Phi^{(\lambda\mu)}) + E(\mathbf{Z}^{(\alpha\beta)},\mathbf{U}^{(\lambda\mu)})] \\
&\quad - D(\mathbf{Z}^{(\alpha\beta)},\Phi^{(\lambda\mu)}) - J(\mathbf{Z}^{(\alpha\beta)},\Phi^{(\lambda\mu)}) + H(W^{(\alpha\beta)},X^{(\lambda\mu)}) \\
t\tilde{D}^{\alpha\beta\lambda\mu}_{:} &= \{D^{\alpha\beta\lambda\mu}\} - A(\mathbf{U}^{(\lambda\mu)},\mathbf{U}^{(\alpha\beta)}) - [E(\mathbf{U}^{(\lambda\mu)},\Phi^{(\alpha\beta)}) + E(\Phi^{(\lambda\mu)},\mathbf{U}^{(\alpha\beta)})] \\
&\quad - D(\Phi^{(\lambda\mu)},\Phi^{(\alpha\beta)}) - J(\Phi^{(\lambda\mu)},\Phi^{(\alpha\beta)}) + H(W^{(\alpha\beta)},W^{(\lambda\mu)}).
\end{aligned} \tag{28}$$

By symmetry of the bilinear forms A, H, J, D and E, the stiffnesses given by eqns (28) satisfy the symmetry conditions (11).

We shall show that the energy of the effective plate,

$$\tilde{W} = \tfrac{1}{2}(\tilde{A}^{\alpha\beta\lambda\mu}_{:}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \tilde{E}^{\alpha\beta\lambda\mu}_{:}\gamma_{\alpha\beta}\kappa_{\lambda\mu} + \tilde{F}^{\alpha\beta\lambda\mu}_{:}\kappa_{\alpha\beta}\gamma_{\lambda\mu} + \tilde{D}^{\alpha\beta\lambda\mu}_{:}\kappa_{\alpha\beta}\kappa_{\lambda\mu}), \tag{29}$$

is positive definite. To this end, we define

$$\bar{\gamma}_{\alpha\beta} = a^{\lambda\mu}_{\alpha\beta}(\mathbf{y})\gamma_{\lambda\mu} + e^{\lambda\mu}_{\alpha\beta}(\mathbf{y})\kappa_{\lambda\mu},\quad 2\bar{\gamma}_{\alpha3} = a^{\lambda\mu}_{\alpha}(\mathbf{y})\gamma_{\lambda\mu} + e^{\lambda\mu}_{\alpha}(\mathbf{y})\kappa_{\lambda\mu}, \tag{30}$$

where

$$\begin{aligned}
a^{\lambda\mu}_{\alpha\beta} &= \delta^{\lambda}_{(\alpha}\delta^{\mu}_{\beta)} + T^{(\lambda\mu)}_{(\alpha|\beta)} + \hat{y}_3 Z^{(\lambda\mu)}_{(\alpha|\beta)},\quad a^{\lambda\mu}_{\alpha} = Z^{(\lambda\mu)}_{\alpha} + X^{(\lambda\mu)}_{|\alpha} \\
e^{\lambda\mu}_{\alpha\beta} &= \hat{y}_3\delta^{\lambda}_{(\alpha}\delta^{\mu}_{\beta)} + U^{(\lambda\mu)}_{(\alpha|\beta)} + \hat{y}_3\Phi^{(\lambda\mu)}_{(\alpha|\beta)},\quad e^{\lambda\mu}_{\alpha} = \Phi^{(\lambda\mu)}_{\alpha} + W^{(\lambda\mu)}_{|\alpha}.
\end{aligned} \tag{31}$$

By constructing appropriate identities from equations of $(\hat{P}^{\alpha}_{\mathrm{loc}})$, one can prove that

$$\tilde{W} = \tfrac{1}{2}\langle\tilde{C}^{\alpha\beta\lambda\mu}\bar{\gamma}_{\alpha\beta}\bar{\gamma}_{\lambda\mu}\rangle + 2\langle C^{\alpha3\beta3}\bar{\gamma}_{\alpha3}\bar{\gamma}_{\beta3}\rangle. \tag{32}$$

Positive definiteness of (C^{ijkl}) implies that $(\tilde{C}^{\alpha\beta\lambda\mu})$, $(C^{\alpha3\beta3})$ are positive definite. Hence $\tilde{W}$ is non-negative. Assume that $\tilde{W} = 0$. Then $\bar{\gamma}_{\alpha\beta} = 0$ and $\bar{\gamma}_{\alpha3} = 0$ for all $\mathbf{y} \in \mathscr{Y}$ and consequently $\{\bar{\gamma}_{\alpha\beta}\} = 0$, $\{\bar{\gamma}_{\alpha3}\} = 0$. Thus, $\gamma_{\alpha\beta} + \hat{y}_3\kappa_{\alpha\beta} = 0$ for all $y_3 \in (c^-, c^+)$, which implies $\gamma_{\alpha\beta} = 0$, $\kappa_{\alpha\beta} = 0$. Thus, $\tilde{W}$ given by eqn (29) is a positive definite quadratic form and hence a constant $c > 0$ exists such that inequality (12) holds.

By virtue of the properties of symmetry and positive definiteness proved above, the static problem for the effective plate with stiffnesses (27) is well posed. This holds irrespective of introducing the shear correction to the definitions (21b) of shearing stiffnesses $H^{\alpha\beta}$.

4. ORTHOTROPIC PLATES OF STIFFNESSES PERIODICALLY VARYING ALONG ONE ORTHOTROPY AXIS

In the case of unidirectional periodicity the algorithm presented in Section 3 enables one to express the effective stiffnesses of indices $(\alpha\alpha\beta\beta)$ by closed formulae and, in certain specific cases, to find closed formulae for the stiffnesses of indices (1212). Because of the peculiar role played in the theory of plate optimization by ribbed plates of finite rank [see Lur'e and Cherkaev (1986) and Lipton (1994)], the analytical formulae for the relevant effective stiffnesses are of vital interest in making the plate optimization problems well posed. Thus, the formulae for effective stiffnesses deserve a detailed derivation.

Assume that both elastic moduli and functions $c^{\pm}(\cdot)$ are $Y_1 = a$-periodic in y_1 and constant in y_2, the y_α axes being orthotropy axes of the material of the plate. The analytical methods used below need reformulation of the $(\hat{P}^{\gamma}_{\text{loc}})$ problems to their strong form.

4.1. *Strong formulation of* $(\hat{P}^1_{loc})$

The strong formulation of $(\hat{P}^1_{\text{loc}})$ will only be given for the considered case of unidirectional periodicity. The unknown functions $T^{(\alpha\beta)}_{\lambda}, Z^{(\alpha\beta)}_{\lambda}, X^{(\alpha\beta)}$ are assumed to be of $C^1(0, a)$ class. The variational equation (24) implies

$$N^{11}_{1(\alpha\beta)|1} = 0, \quad N^{12}_{1(\alpha\beta)|1} = 0, \quad Q^{1}_{1(\alpha\beta)|1} = 0, \tag{33a–e}$$
$$-M^{11}_{1(\alpha\beta)|1} + Q^{1}_{1(\alpha\beta)} = 0, \quad -M^{12}_{1(\alpha\beta)|1} + Q^{2}_{1(\alpha\beta)} = 0,$$

where [cf. eqns (23)]

$$N^{\gamma\gamma}_{1(\alpha\beta)} = \mathrm{A}^{\gamma\gamma 11} T^{(\alpha\beta)}_{1|1} + E^{\gamma\gamma 11} Z^{(\alpha\beta)}_{1|1} + A^{\gamma\gamma\alpha\beta}, \quad N^{12}_{1(\alpha\beta)} = A^{1212} T^{(\alpha\beta)}_{2|1} + E^{1212} Z^{(\alpha\beta)}_{2|1} + A^{12\alpha\beta}$$
$$Q^{1}_{1(\alpha\beta)} = H^{11}(X^{(\alpha\beta)}_{|1} + Z^{(\alpha\beta)}_{1}), \quad Q^{2}_{1(\alpha\beta)} = H^{22} Z^{(\alpha\beta)}_{2},$$
$$M^{\gamma\gamma}_{1(\alpha\beta)} = E^{\gamma\gamma 11} T^{(\alpha\beta)}_{1|1} + D^{\gamma\gamma 11} Z^{(\alpha\beta)}_{1|1} + E^{\gamma\gamma\alpha\beta}, \quad M^{12}_{1(\alpha\beta)} = E^{1212} T^{(\alpha\beta)}_{2|1} + D^{1212} Z^{(\alpha\beta)}_{2|1} + E^{12\alpha\beta}. \tag{34a–f}$$

Equations (33) and (34), along with the following periodicity conditions,

$$T^{(\alpha\beta)}_{\lambda}(0) = T^{(\alpha\beta)}_{\lambda}(a), \quad Z^{(\alpha\beta)}_{\lambda}(0) = Z^{(\alpha\beta)}_{\lambda}(a), \quad X^{(\alpha\beta)}(0) = X^{(\alpha\beta)}(a) \tag{35}$$

$$N^{1\gamma}_{1(\alpha\beta)}(0) = N^{1\gamma}_{1(\alpha\beta)}(a), \quad M^{1\gamma}_{1(\alpha\beta)}(0) = M^{1\gamma}_{1(\alpha\beta)}(a), \quad Q^{1}_{1(\alpha\beta)}(0) = Q^{1}_{1(\alpha\beta)}(a), \tag{36}$$

constitute the strong formulation of the problem.

4.2. *Solution to* $(\hat{P}^1_{loc})$

Note that the problem (33)–(36) splits up into two problems involving the unknowns:

$$\text{(a)}\ T^{(\alpha\beta)}_1, Z^{(\alpha\beta)}_1, X^{(\alpha\beta)}; \quad \text{(b)}\ T^{(\alpha\beta)}_2, Z^{(\alpha\beta)}_2.$$

Let us start with problem (a). Function $X^{(\alpha\beta)}$ will not be necessary [cf. eqns (27) and (34)]. We are looking for only the derivatives $T^{(\alpha\beta)}_{1|1}, Z^{(\alpha\beta)}_{1|1}$. By eqns (33) we infer that

$$N^{11}_{1(\alpha\beta)} = A^{(\alpha\beta)}_1, \quad Q^{1}_{1(\alpha\beta)} = A^{(\alpha\beta)}_3, \quad M^{11}_{1(\alpha\beta)} = A^{(\alpha\beta)}_3 y_1 + A^{(\alpha\beta)}_2, \tag{37a–c}$$

where $A^{(\alpha\beta)}_j$ are constants. The second condition in eqns (36) implies $A^{(\alpha\beta)}_3 = 0$. We equate the first expression of eqns (34) for $\gamma = 1$ to constant $A^{(\alpha\beta)}_1$ and the fourth equation of (34) for $\gamma = 1$ to the constant $A^{(\alpha\beta)}_2$. We find

$$T_{1|1}^{(\alpha\beta)} = \frac{1}{G^2}[A_1^{(\alpha\beta)}D^{1111} - A_2^{(\alpha\beta)}E^{1111} + (E^{1111}E^{11\alpha\beta} - D^{1111}A^{11\alpha\beta})]$$

$$Z_{1|1}^{(\alpha\beta)} = \frac{1}{G^2}[-A_1^{(\alpha\beta)}E^{1111} + A_2^{(\alpha\beta)}A^{1111} + (E^{1111}A^{11\alpha\beta} - A^{1111}E^{11\alpha\beta})], \tag{38}$$

where

$$G = (A^{1111}D^{1111} - (E^{1111})^2)^{1/2}.$$

The braces $\{\cdot\}$ mean here $a^{-1}\int_0^a (\cdot)\,dy_1$. Functions $T_1^{(\alpha\beta)}, Z_1^{(\alpha\beta)}$ are a-periodic, hence $\{T_{1|1}^{(\alpha\beta)}\} = 0$, $\{Z_{1|1}^{(\alpha\beta)}\} = 0$. Thus, averaging both sides of eqns (38) leads to an equation that interrelates the constants $A_\gamma^{(\alpha\beta)}$:

$$\mathbf{f}\begin{bmatrix} A_1^{(\alpha\beta)} \\ A_2^{(\alpha\beta)} \end{bmatrix} = \begin{bmatrix} f_{10}^{(\alpha\beta)} \\ f_{20}^{(\alpha\beta)} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \tag{39}$$

where

$$f_{11} = \{D^{1111}/G^2\}, \quad f_{12} = f_{21} = -\{E^{1111}/G^2\}, \quad f_{22} = \{A^{1111}/G^2\} \tag{40}$$

and

$$f_{10}^{(\alpha\beta)} = \{(D^{1111}A^{11\alpha\beta} - E^{1111}E^{11\alpha\beta})/G^2\},$$
$$f_{20}^{(\alpha\beta)} = \{(A^{1111}E^{11\alpha\beta} - E^{1111}A^{11\alpha\beta})/G^2\}. \tag{41}$$

Thus, the constants $A_\gamma^{(\alpha\beta)}$ are expressed as follows:

$$A_1^{(\alpha\beta)} = (f_{22}f_{10}^{(\alpha\beta)} - f_{12}f_{20}^{(\alpha\beta)})g^2, \quad A_2^{(\alpha\beta)} = (-f_{12}f_{10}^{(\alpha\beta)} + f_{11}f_{20}^{(\alpha\beta)})g^2, \tag{42}$$

where

$$g = (\det \mathbf{f})^{-1/2} = [f_{11}f_{22} - (f_{12})^2]^{-1/2}. \tag{43}$$

In view of the symmetry of matrix $\mathbf{f}$, the following identities hold:

$$A_1^{(\alpha\beta)}f_{10}^{(\lambda\mu)} + A_2^{(\alpha\beta)}f_{20}^{(\lambda\mu)} = A_1^{(\lambda\mu)}f_{10}^{(\alpha\beta)} + A_2^{(\lambda\mu)}f_{20}^{(\alpha\beta)}. \tag{44}$$

Some of these identities can be additionally simplified by taking into account that

$$f_{10}^{(11)} = 1, \quad f_{20}^{(11)} = 0. \tag{45}$$

Equations (38)–(43) determine derivatives of functions $T_1^{(\alpha\beta)}, Z_1^{(\alpha\beta)}$. These derivatives vanish for $\alpha \neq \beta$, which is a consequence of the orthotropy of the plate material.

Let us proceed to determination of functions $T_2^{(\alpha\beta)}, Z_2^{(\alpha\beta)}$. Because of orthotropy $T_2^{(\alpha\alpha)} = 0, Z_2^{(\alpha\alpha)} = 0$. Functions: $T_2^{(12)}, Z_2^{(12)}$ are unknown. It turns out that in a general case one cannot express these functions by closed formulae. Let us consider two specific cases separately:

(A) Stiffnesses A^{1212}, E^{1212}, D^{1212} are of $C^1(0,a)$ class.
(B) These stiffnesses are piecewise constant.

Consider case (A). On using eqns (34b, d, f), eqns (33b, e) can be re-written in the form

$$A^{1212}T^{(12)}_{2|1}+E^{1212}Z^{(12)}_{2|1}+A^{1212} = C_1,$$

$$E^{1212}T^{(12)}_{2|1}+D^{1212}Z^{(12)}_{2|1}+E^{1212} = \int_0^{y_1} H^{22}Z^{(12)}_{2|1}\,dy_1+C_2, \quad (46)$$

where C_α are constants. The periodicity conditions (35), (36) for $\gamma = 2$ can be expressed as follows:

$$\{H^{22}Z_2^{(12)}\} = 0, \quad Z_2^{(12)}(0) = Z_2^{(12)}(a), \quad T_2^{(12)}(0) = T_2^{(12)}(a). \quad (47)$$

Let $Z_2^{(12)} = C_1Z(y_1)$. Function Z satisfies the following ordinary differential equation:

$$LZ = -(E^{1212}/A^{1212})_{|1}, \quad (48)$$

where

$$L(\cdot) = \frac{d}{dy_1}\left[P(y_1)\frac{d}{dy_1}(\cdot)\right]-H^{22}(y_1)(\cdot)$$

$$P(y_1) = D^{1212}(y_1)-\frac{[E^{1212}(y_1)]^2}{A^{1212}(y_1)}. \quad (49)$$

Solution Z is uniquely determined by the conditions

$$Z(0) = Z(a), \quad \{H^{22}Z\} = 0. \quad (50)$$

We shall find the constants C_α. Let us rearrange eqn (42) into the form

$$T^{(12)}_{2|1} = G^{-2}\left[D^{1212}(C_1-A^{1212})-E^{1212}\left(C_2+C_1\int_0^{y_1} H^{22}Z\,dy_1-E^{1212}\right)\right] \quad (51)$$

$$Z^{(12)}_{2|1} = G^{-2}\left[-E^{1212}(C_1-A^{1212})+A^{1212}\left(C_2+C_1\int_0^{y_1} H^{22}Z\,dy_1-E^{1212}\right)\right].$$

We remember that

$$\{T^{(12)}_{2|1}\} = 0, \quad \{Z^{(12)}_{2|1}\} = 0. \quad (52)$$

On averaging both sides of eqns (51) and using the equalities (52), one finds algebraic equations for constants C_α.

Case (B). For simplicity we assume the following stepwise variation of moduli X^{1212}, $X = A, E, D$:

$$X^{1212} = \begin{cases} X_1^{1212}, & y_1 \in I_1 \\ X_2^{1212}, & y_1 \in I_2 \end{cases}, \quad X = A, E, D,$$

where $I_1 = (0,b_1)$, $I_2 = (b_1, a)$, $b_2 = a-b_1$. Solutions to eqns (46) are predicted as follows:

$$T_2^{(12)} = \begin{cases} u_1 \\ u_2 \end{cases}, \quad Z_2^{(12)} = \begin{cases} v_1 \\ v_2 \end{cases}, \quad \text{for} \begin{cases} y_1 \in I_1 \\ y_1 \in I_2 \end{cases}.$$

We eliminate the unknowns u_α and find

$$\frac{d^2 v_\sigma}{dy_1^2} - (\alpha_\sigma)^2 v = 0, \tag{53}$$

where

$$\alpha_\sigma = (H_\sigma^{22}/P_\sigma)^{1/2}, \quad P_\sigma = D_\sigma^{1212} - (E_\sigma^{1212})^2/A_\sigma^{1212}. \tag{54}$$

Let

$$y_1 = b_1\xi, \quad \lambda = b_1\alpha_1, \quad \lambda\sigma = b_1\alpha_2, \quad \sigma = \alpha_2/\alpha_1. \tag{55}$$

Then

$$\frac{d^2 v_1}{d\xi^2} - \lambda^2 v_1 = 0, \quad \frac{d^2 v_2}{d\xi^2} - \lambda^2\sigma^2 v_2 = 0. \tag{56}$$

Solutions v_α are represented in the form

$$\begin{aligned} v_1 &= C_1 e^{-\lambda\xi} + B_1 e^{-\lambda(1-\xi)}, \quad \xi \in (0,1) \\ v_2 &= C_2 e^{-\lambda\sigma(\xi-1)} + B_2 e^{-\lambda\sigma(\omega-\xi)}, \quad \xi \in (1,\omega), \end{aligned} \tag{57}$$

where $\omega = a/b_1$. Functions u_α are of the form

$$\begin{aligned} u_1 &= -\mu_1 v_1(\xi) + bK_1\xi + F_1, \quad \xi \in (0,1) \\ u_2 &= -\mu_2 v_2(\xi) + bK_2\xi + F_2, \quad \xi \in (1,\omega), \end{aligned} \tag{58}$$

where $\mu_\alpha = E_\alpha^{1212}/A_\alpha^{1212}$. Eight constants C_α, B_α, K_α, F_α can be found from the periodicity conditions:

$$v_1(0) = v_2(\omega), \quad u_1(0) = u_2(\omega), \quad M_{1(12)}^{(12)}(0) = M_{1(12)}^{(12)}(\omega), \quad N_{1(12)}^{(12)}(0) = N_{1(12)}^{(12)}(\omega) \tag{59}$$

and from switching conditions at point $y_1 = b_1$ or $\xi = 1$

$$\begin{aligned} &v_1(1) = v_2(1), \quad u_1(1) = u_2(1) \\ &M_{1(12)}^{(12)}(1-0) = M_{1(12)}^{(12)}(1+0), \quad N_{1(12)}^{(12)}(1-0) = N_{1(12)}^{(12)}(1+0). \end{aligned} \tag{60}$$

Taking into account definitions (34b,e) and relations (58), quantities $N_{1(12)}^{12}, M_{1(12)}^{12}$ can be represented in the form

$$N^{12}_{1(12)} = \begin{cases} A_1^{1212}(1+K_1), & \xi \in (0,1) \\ A_2^{1212}(1+K_2), & \xi \in (1,\omega) \end{cases} \tag{61}$$

$$M^{12}_{1(12)} = \begin{cases} b_1^{-1} P_1 \dfrac{dv_1}{d\xi} + E_1^{1212}(1+K_1), & \xi \in (0,1) \\ b_1^{-1} P_2 \dfrac{dv_2}{d\xi} + E_2^{1212}(1+K_2), & \xi \in (1,\omega). \end{cases} \tag{62}$$

We report the final formulae for the integration constants. Let

$$N = \lambda[(1-\kappa_1\mu_1)(1+e^{-\lambda})(1-e^{-\lambda\sigma(\omega-1)}) + \delta\sigma(1-\kappa_2\mu_2)(1-e^{-\lambda})(1+e^{-\lambda\sigma(\omega-1)})] \tag{63}$$

$$M = 2(\mu_1-\mu_2)(\delta\kappa_2-\kappa_1\varepsilon)(1-e^{-\lambda})(1-e^{-\lambda\sigma(\omega-1)}) - (\varepsilon+\omega-1)N, \tag{64}$$

where

$$\varepsilon = A_2^{1212}/A_1^{1212}, \quad \delta = D_2^{1212}/D_1^{1212}, \quad \kappa_\alpha = E_\alpha^{1212}/D_\alpha^{1212}. \tag{65}$$

Constants K_α, B_α are given by

$$K_1 = -\frac{\omega\varepsilon N}{M} - 1, \quad K_2 = -\frac{\omega N}{M} - 1 \tag{66}$$

$$B_1/b_1 = \omega(\kappa_1\varepsilon - \delta\kappa_2)(1-e^{-\lambda\sigma(\omega-1)})M^{-1}, \quad B_2/b_1 = -\omega(\kappa_1\varepsilon-\delta\kappa_2)(1-e^{-\lambda})M^{-1} \tag{67}$$

and $C_\alpha = -B_\alpha$. Constant F_1+F_2 cannot be determined. Constant F_1-F_2 will not be used in the sequel and that is why it is not reported.

The above results make it possible to express the effective stiffnesses A_h^{1212}, E_h^{1212} by closed formulae. The formulae will be given in a later part of the paper.

4.3. *Strong formulation of the* $(\hat{P}^2_{loc})$ *problem*

The formulation will be given only for the case of periodicity in one direction, y_1. The unknown fields $U_\lambda^{(\alpha\beta)}, \Phi_\lambda^{(\alpha\beta)}, W^{(\alpha\beta)}$ are dependent only on $y_1 \in [0, a]$. The form of the strong formulation follows from similarity between problems $(\hat{P}^1_{loc})$ and $(\hat{P}^2_{loc})$ (see Section 4.1). We replace:

$$T_\lambda^{(\alpha\beta)} \to U_\lambda^{(\alpha\beta)}, \quad Z_\lambda^{(\alpha\beta)} \to \Phi_\lambda^{(\alpha\beta)}, \quad X^{(\alpha\beta)} \to W^{(\alpha\beta)} \tag{68}$$

$$N^{1\gamma}_{1(\alpha\beta)} \to N^{1\gamma}_{2(\alpha\beta)}, \quad M^{1\gamma}_{1(\alpha\beta)} \to M^{1\gamma}_{2(\alpha\beta)}, \quad Q^{\gamma}_{1(\alpha\beta)} \to Q^{\gamma}_{2(\alpha\beta)}. \tag{69}$$

Moreover, the free terms $A^{\gamma\gamma\alpha\beta}$, $A^{12\alpha\beta}$ in eqns (34a, b) should be replaced with $E^{\gamma\gamma\alpha\beta}$, $E^{12\alpha\beta}$ and the free terms $E^{\gamma\gamma\alpha\beta}$, $E^{12\alpha\beta}$ in eqns (34e, f) with $D^{\gamma\gamma\alpha\beta}$, $D^{12\alpha\beta}$ [see eqn (23)].

4.4. *Solution of the* $(\hat{\mathrm{P}}^2_{loc})$ *problem*

4.4.1. *Fields* $\mathrm{U}_1^{(\alpha\beta)}, \Phi_1^{(\alpha\beta)}$. Proceeding similarly as in Section 4.2, we find

$$U_{1|1}^{(\alpha\beta)} = \frac{1}{G^2}[B_1^{(\alpha\beta)}D^{1111} - B_2^{(\alpha\beta)}E^{1111} + (E^{1111}D^{11\alpha\beta} - D^{1111}E^{11\alpha\beta})],$$

$$\Phi_{1|1}^{(\alpha\beta)} = \frac{1}{G^2}[-B_1^{(\alpha\beta)}E^{1111} + B_2^{(\alpha\beta)}A^{1111} + (E^{1111}E^{11\alpha\beta} - A^{1111}D^{11\alpha\beta})], \tag{70}$$

where $B_\alpha^{(\lambda\mu)}$ are solutions to the following set of algebraic equations:

$$\mathbf{f}\begin{bmatrix} B_1^{(\alpha\beta)} \\ B_2^{(\alpha\beta)} \end{bmatrix} = \begin{bmatrix} g_{10}^{(\alpha\beta)} \\ g_{20}^{(\alpha\beta)} \end{bmatrix}, \tag{71}$$

where $\mathbf{f}$ has been defined by eqns (39) and

$$g_{10}^{(\alpha\beta)} = \{(D^{1111}E^{11\alpha\beta} - E^{1111}D^{11\alpha\beta})/G^2\}, \quad g_{20}^{(\alpha\beta)} = \{(A^{1111}D^{11\alpha\beta} - E^{1111}E^{11\alpha\beta})/G^2\} \tag{72}$$

and

$$g_{10}^{(11)} = 0, \quad g_{20}^{(11)} = 1. \tag{73}$$

In view of the symmetry of matrix $\mathbf{f}$, the following identities hold:

$$B_1^{(\alpha\beta)}g_{10}^{(\lambda\mu)} + B_2^{(\alpha\beta)}g_{20}^{(\lambda\mu)} = B_1^{(\lambda\mu)}g_{10}^{(\alpha\beta)} + B_2^{(\lambda\mu)}g_{20}^{(\alpha\beta)},$$

$$A_1^{(\alpha\beta)}g_{10}^{(\lambda\mu)} + A_2^{(\alpha\beta)}g_{20}^{(\lambda\mu)} = B_1^{(\lambda\mu)}f_{10}^{(\alpha\beta)} + B_2^{(\lambda\mu)}f_{20}^{(\alpha\beta)}, \tag{74a,b}$$

where constants $A_i^{(\alpha\beta)}$ are given by eqns (39) or (42). In view of relations (45) and (73), some of these identities assume a simple form, e.g.

$$A_2^{(11)} = B_1^{(11)}, \tag{75}$$

$$A_2^{(22)} = B_1^{(11)}f_{10}^{(22)} + B_2^{(11)}f_{20}^{(22)}. \tag{76}$$

Note that the functions $U_1^{(12)}, \Phi_1^{(12)}$ are constant. Functions $W^{(\alpha\beta)}$ will not be necessary in the sequel.

4.4.2. *Fields* $\mathrm{U}_2^{(\alpha\beta)}, \Phi_2^{(\alpha\beta)}$. If $\alpha = \beta$, these functions are constant. Thus, the problem is reduced to finding $U_2^{(12)}, \Phi_2^{(12)}$. Let us consider independently two cases of distribution of stiffnesses along $(0,a)$.

(A) Stiffnesses A^{1212}, E^{1212}, D^{1212} are of $C^1(0,a)$ class. Despite some similarities with problem $(\hat{P}^1_{loc})$, the problem considered here is more difficult. The starting point is the set of equations:

$$A^{1212}U_{2|1}^{(12)} + E^{1212}(\Phi_{2|1}^{(12)} + 1) = D_1, \quad [E^{1212}U_{2|1}^{(12)} + D^{1212}(\Phi_{2|1}^{(12)} + 1)]_{|1} = H^{22}\Phi_2^{(12)}, \tag{77a,b}$$

where D_1 is a constant. The periodicity conditions imply

$$\int_0^a H^{22}\Phi_2^{(12)}\,\mathrm{d}y_1 = 0, \quad \Phi_2^{(12)}(0) = \Phi_2^{(12)}(a), \quad U_2^{(12)}(0) = U_2^{(12)}(a). \tag{78a–c}$$

We eliminate $U_2^{(12)}$ and find

$$L\Phi_2^{(12)} = -P_{|1} - D_1(E^{1212}/A^{1212})_{|1}. \tag{79}$$

Thus,

$$\Phi_2^{(12)} = \Phi_a + D_1\Phi_b + D_3\Phi_1 + D_4\Phi_2, \tag{80}$$

where Φ_a, Φ_b are arbitrary specific integrals of

$$L\Phi_a = -P_{|1}, \quad L\Phi_b = -(E^{1212}/A^{1212})_{|1},$$

whereas Φ_α are linearly independent solutions to the equation: $\mathrm{L}\Phi_\alpha = 0$. By eqn (77a) one can express $U_2^{(12)}$ in terms of the constants D_1, D_3, D_4 and D_2. The latter constant cancels from both sides of eqn (78c). Thus, the three conditions (78) yield three equations for D_1, D_3 and D_4.

(B) Stiffnesses A^{1212}, E^{1212}, D^{1212} are piecewise constant. We proceed as in Section 4.2.B. We assume the same distribution of stiffnesses along the $(0,a)$ interval. The solutions are predicted in the following form:

$$U_2^{(12)} = \begin{cases} u_1(\xi) \\ u_2(\xi) \end{cases}, \quad \Phi_2^{(12)} = \begin{cases} \varphi_1(\xi) \\ \varphi_2(\xi) \end{cases}, \quad \text{for} \quad \begin{cases} \xi\in(0,1) \\ \xi\in(1,\omega) \end{cases}, \tag{81}$$

where

$$u_1 = B_1[-\mathrm{e}^{-\lambda\xi} + \mathrm{e}^{-\lambda(1-\xi)}], \quad u_2 = B_2[-\mathrm{e}^{-\lambda\sigma(\xi-1)} + \mathrm{e}^{-\lambda\sigma(\omega-\xi)}] \tag{82}$$

$$\varphi_\alpha = -\mu_\alpha u_\alpha + bK_\alpha\xi + F_\alpha.$$

We have at once assumed that $B_\alpha = -C_\alpha$ (cf. Section 4.2). The stress resultants $N_{2(12)}^{(12)}$, $M_{2(12)}^{(12)}$ are given by

$$N_{2(12)}^{12} = \begin{cases} A_1^{1212}(\mu_1 + K_1), & \xi\in(0,1) \\ A_2^{1212}(\mu_2 + K_2), & \xi\in(1,\omega) \end{cases} \tag{83}$$

$$M_{2(12)}^{12} = \begin{cases} b_1^{-1}P_1\dfrac{\mathrm{d}\varphi_1}{\mathrm{d}\xi} + D_1^{1212}(1+\kappa_1K_1), & \xi\in(0,1) \\ b_1^{-1}P_2\dfrac{\mathrm{d}\varphi_2}{\mathrm{d}\xi} + D_2^{1212}(1+\kappa_2K_2), & \xi\in(1,\omega). \end{cases} \tag{84}$$

By periodicity and switching conditions for $\xi = 1$ one can find the constants B_α and K_α. These formulae will not be reported.

4.5. *Formulae for effective stiffnesses*

The effective stiffnesses of the original $\varepsilon\mathscr{Y}$-periodic plate are given by eqns (9). For the sake of simplicity we now assume $\varepsilon = 1$ and the abbreviation hom will be replaced with h. Thus, the effective stiffnesses are denoted by

$$X_h^{\alpha\beta\lambda\mu} = t\tilde{X}^{\alpha\beta\lambda\mu}, \quad \tilde{X} = \tilde{A}, \tilde{F}, \tilde{E}, \tilde{D}.$$

In view of the assumption of orthotropy, the only non-zero components of $X_h^{\alpha\beta\lambda\mu}$ are those for which: $\alpha+\beta+\lambda+\mu$ assume values of 4, 6 or 8. Below we shall deal only with such components.

(A) Stiffnesses $A_h^{\alpha\alpha\beta\beta}$. According to eqns (27a) and (37a) we have

$$A_h^{\alpha\alpha 11} = A_1^{(\alpha\alpha)}, \tag{85}$$

where $A_1^{(\alpha\alpha)}$ are determined by eqns (42). In particular,

$$A_h^{1111} = \left\{\left(\frac{g}{G}\right)^2 A^{1111}\right\}. \tag{86}$$

The stiffnesses $A_h^{\alpha\alpha 22}$ can be determined by eqns (27a), (34a) and (38). We find

$$A_h^{1122} = A_1^{(11)}f_{10}^{(22)} + A_2^{(11)}f_{20}^{(22)},$$
$$A_h^{2222} = \{A^{2222}\} + A_1^{(22)}f_{10}^{(22)} + A_2^{(22)}f_{20}^{(22)} - A_3, \tag{87}$$

where

$$A_3 = \{[(A^{1122})^2 D^{1111} - 2E^{1111}E^{1122}A^{1122} + (E^{1122})^2 A^{1111}]/G^2\}. \tag{88}$$

From the identity (44) for $(\alpha\beta) = (11)$, $(\lambda\mu) = (22)$, one can infer the equality

$$A_h^{1122} = A_h^{2211}, \tag{89}$$

which has been proved previously in a general case [see eqns (11a) and (27a)]. Thus, $A_h^{1122} = A_1^{(22)}$.

(B) Stiffnesses $F_h^{\alpha\alpha\beta\beta}$. Using eqns (27b), (34e) and (38), one finds

$$F_h^{\alpha\alpha 11} = A_2^{(\alpha\alpha)}, \quad F_h^{1122} = A_1^{(11)}g_{10}^{(22)} + A_2^{(11)}g_{20}^{(22)},$$
$$F_h^{2222} = \{E^{2222}\} + A_1^{(22)}g_{10}^{(22)} + A_2^{(22)}g_{20}^{(22)} + A_4, \tag{90}$$

$$A_4 = \{[E^{2211}(E^{1111}E^{1122} - D^{1111}A^{1122}) + D^{2211}(E^{1111}A^{1122} - A^{1111}E^{1122})]/G^2\}. \tag{91}$$

(C) Stiffnesses $E_h^{\alpha\alpha\beta\beta}$. Using definition (27c) and the approximate solutions found in Section 4.4 one finds

$$E_h^{\alpha\alpha 11} = B_1^{(\alpha\alpha)}, \quad E_h^{1122} = B_1^{(11)}f_{10}^{(22)} + B_2^{(11)}f_{20}^{(22)},$$
$$E_h^{2222} = \{E^{2222}\} + B_1^{(22)}f_{10}^{(22)} + B_2^{(22)}f_{20}^{(22)} + A_5, \tag{92}$$

$$A_5 = \{[A^{2211}(E^{1111}D^{1122} - D^{1111}E^{1122}) + E^{2211}(E^{1111}E^{1122} - A^{1111}D^{1122})]/G^2\}. \tag{93}$$

Note that $A_4 = A_5$. Moreover, using the identity (74b) one can prove that

$$E_h^{\alpha\alpha\beta\beta} = F_h^{\beta\beta\alpha\alpha}, \tag{94}$$

[cf. eqn (11b)]. Of formulae (92), the simplest one is the following:

$$E_h^{1111} = F_h^{1111} = g^2\{E^{1111}/G^2\}. \tag{95}$$

(D) Stiffnesses $D_h^{\alpha\alpha\beta\beta}$. We make use of definitions (27d) and the results of Section 4.4. We find

$$D_h^{\alpha\alpha 11} = B_2^{(\alpha\alpha)}, \quad D_h^{1122} = B_1^{(11)}g_{10}^{(22)} + B_2^{(11)}g_{20}^{(22)},$$

$$D_h^{2222} = \{D^{2222}\} + B_1^{(22)}g_{10}^{(22)} + B_2^{(22)}g_{20}^{(22)} + A_6, \tag{96}$$

$$A_6 = \{[E^{2211}(E^{1111}D^{1122} - D^{1111}E^{1122}) + D^{2211}(E^{1111}E^{1122} - A^{1111}D^{1122})]/G^2\}. \tag{97}$$

Using the identities (74d) one can confirm the equality $D_h^{1122} = D_h^{2211}$ proved in Section 3.2 in a general case. Thus, $D_h^{1122} = B_2^{(22)}$.

(E) Stiffnesses $A_h^{1212}, F_h^{1212} = E_h^{1212}, D_h^{1212}$.

(Ei) Case of a smooth distribution of stiffnesses. Using definitions (27a,b), eqns (34b,f) and formulae (46), one finds

$$A_h^{1212} = C_1, F_h^{1212} = C_2 + C_1\left\{\int_0^{y_1} H^{22} Z \, \mathrm{d}y_1\right\}. \tag{98}$$

The stiffnesses E_h^{1212}, D_h^{1212} can be found by definitions (27c,d), eqns (23c,d) and (77a):

$$E_h^{1212} = D_1, D_h^{1212} = \{P(1 + \Phi_{2|1}^{(12)})\} + D_1\{E^{1212}/A^{1212}\}, \tag{99}$$

where one should insert eqn (80). The identity $E_h^{1212} = F_h^{1212}$ holds, but cannot be easy inferred from the equations given above.

(Eii) Case of piecewise varying stiffnesses. Stiffnesses A_h^{1212}, F_h^{1212} are determined by eqns (27a,b), (34b,f), (61) and (62). On using these equations one finds

$$A_h^{1212} = A_1^{1212}(1 + K_1) \tag{100}$$

$$F_h^{1212} = \gamma_1 E_1^{1212}(1 + K_1) + \gamma_2 E_2^{1212}(1 + K_2) + \frac{P_1 - P_2}{a}[v_1(1) - v_1(0)], \tag{101}$$

where $\gamma_\alpha = b_\alpha/a$. The equations given above can be expressed in a form depending explicitly upon the parameters characterizing properties of the cell of periodicity. Considering results (66) and (67) one can rearrange expressions (100) and (101) into the form

$$A_h^{1212} = \left[\frac{\gamma_1}{A_1^{1212}} + \frac{\gamma_2}{A_2^{1212}} + \frac{(\mu_1 - \mu_2)^2}{Z}\right]^{-1}, \tag{102}$$

$$E_h^{1212} = \mu_h A_h^{1212}, \tag{103}$$

where

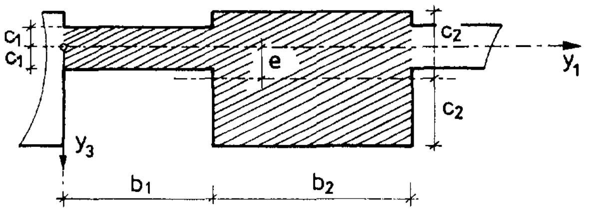

Fig. 1. Periodicity cell $\mathscr{Y}$ of transversely asymmetric plate.

$$\mu_h = \gamma_1\mu_1 + \gamma_2\mu_2 - \frac{(\mu_1-\mu_2)(P_1-P_2)}{Z}, \tag{104}$$

$$Z = \tfrac{1}{2}\chi_1 P_1 \coth\frac{\gamma_1\chi_1}{2} + \tfrac{1}{2}\chi_2 P_2 \coth\frac{\gamma_2\chi_2}{2}, \quad \chi_\sigma = a\alpha_\sigma. \tag{105}$$

In a similar manner, one can find stiffnesses F_h^{1212} and D_h^{1212}. First, we confirm the equality $F_h^{1212} = E_h^{1212}$. Then we find

$$D_h^{1212} = P_h + (\mu_h)^2 A_h^{1212},$$

where

$$P_h = \gamma_1 P_1 + \gamma_2 P_2 - \frac{(P_1-P_2)^2}{Z}. \tag{107}$$

5. EFFECTIVE STIFFNESSES OF PLATES WITH THICKNESS PIECEWISE VARYING IN ONE DIRECTION

Consider a plate with thickness varying in the y_1 direction, as shown in Fig. 1. Assume that the plate material is orthotropic of moduli C^{ijkl} and constant with respect to y_1, y_3. The assumption (2) holds here as well as the definition (3) of reduced moduli relating to the generalized plane stress state.

5.1. *Preliminary computations*

In the case considered the cell $\mathscr{Y}$ is plane (cf. Fig. 1). Its area $|\mathscr{Y}|$ equals

$$|\mathscr{Y}| = 2(c_1 b_1 + c_2 b_2) = \frac{2}{a}(\gamma_1 c_1 + \gamma_2 c_2). \tag{108}$$

The quantity $\langle y_3\rangle$ equals

$$\langle y_3\rangle = \frac{1}{|\mathscr{Y}|}\left[\int_0^{b_1}\int_{-c_1}^{c_1} y_3\,\mathrm{d}y_1\,\mathrm{d}y_3 + \int_{b_1}^{a}\int_{-(c_2-e)}^{c_2+e} y_3\,\mathrm{d}y_1\,\mathrm{d}y_3\right]. \tag{109}$$

Thus,

$$\langle y_3\rangle = \beta_2 e, \quad \beta_2 = \frac{\gamma_2 c_2}{\gamma_1 c_1 + \gamma_2 c_2}. \tag{110}$$

We shall use further the quantity $\beta_1 = 1-\beta_2$. According to eqns (21) one computes the stiffnesses as

$$X^{\alpha\beta\lambda\mu} = \begin{cases} X_1^{\alpha\beta\lambda\mu} & y_1 \in (0, b_1) \\ X_2^{\alpha\beta\lambda\mu} & y_1 \in (b_1, a) \end{cases},$$

where $X = A, E, D$. We find

$$A_\sigma^{\alpha\beta\lambda\mu} = 2c_\sigma \tilde{C}^{\alpha\beta\lambda\mu}, \tag{111}$$

$$E_1^{\alpha\beta\lambda\mu} = -2\beta_2 c_1 e \tilde{C}^{\alpha\beta\lambda\mu}, \quad E_2^{\alpha\beta\lambda\mu} = 2\beta_1 c_2 e \tilde{C}^{\alpha\beta\lambda\mu}, \tag{112}$$

$$D_1^{\alpha\beta\lambda\mu} = [\tfrac{2}{3}(c_1)^3 + 2e^2(\beta_2)^2 c_1]\tilde{C}^{\alpha\beta\lambda\mu}, \quad D_2^{\alpha\beta\lambda\mu} = [\tfrac{2}{3}(c_2)^3 + 2e^2(\beta_1)^2 c_2]\tilde{C}^{\alpha\beta\lambda\mu}. \tag{113}$$

Let

$$P_\sigma^{\alpha\beta\lambda\mu} = D_\sigma^{\alpha\beta\lambda\mu} - (E_\sigma^{\alpha\beta\lambda\mu})^2 / A_\sigma^{\alpha\beta\lambda\mu}, \quad \text{for } A_\sigma^{\alpha\beta\lambda\mu} \neq 0. \tag{114}$$

Then P_σ defined by eqn (54) equals P_σ^{1212}. We can easily find that

$$P_\sigma^{\alpha\beta\lambda\mu} = \tfrac{2}{3}(c_\sigma)^3 \tilde{C}^{\alpha\beta\lambda\mu}. \tag{115}$$

Thus, quantities (115) represent bending stiffnesses referred to middle axes (cf. Fig. 1).

We calculate stiffnesses due to transverse shear

$$H^{\alpha\beta} = \begin{cases} H_1^{\alpha\beta} & y_1 \in (0, b_1) \\ H_2^{\alpha\beta} & y_1 \in (b_1, a) \end{cases}, \tag{116}$$

$$H_\sigma^{\alpha\beta} = 2\kappa c_\sigma C^{\alpha 3\beta 3}, \tag{117}$$

where κ is a transverse shear correction factor, usually assumed as 5/6. We compute further

$$\alpha_\sigma = 2\mu/c_\sigma, \quad \chi_\sigma = a\alpha_\sigma = 2\mu a/c_\sigma, \quad \mu = \tfrac{1}{2}(3\kappa C^{2323}/C^{1212})^{1/2}. \tag{118}$$

In the case of isotropy: $\alpha_\sigma = (3\kappa)^{1/2}/c_\sigma$.

5.2. *Stiffnesses of indices* (1212)

Taking into account of the relation $\mu_1 - \mu_2 = -e$, one finds

$$A_h^{1212}/C^{1212} = \frac{2}{\left(\dfrac{\gamma_1}{c_1} + \dfrac{\gamma_2}{c_2}\right) + 3\dfrac{e^2}{R}}, \tag{119}$$

where

$$R = \tfrac{1}{2}\chi_1 (c_1)^3 \coth(\tfrac{1}{2}\gamma_1\chi_1) + \tfrac{1}{2}\chi_2 (c_2)^3 \coth(\tfrac{1}{2}\gamma_2\chi_2) \tag{120}$$

or $R = 1.5Z/C^{1212}$. In view of the relations

$$\tfrac{1}{2}\chi_\sigma (c_\sigma)^3 = \mu a (c_\sigma)^2, \quad \tfrac{1}{2}\gamma_\sigma\chi_\sigma = \mu b_\sigma / c_\sigma,$$

one can express the quantity R as follows:

$$R = a\mu\left[(c_1)^2\coth\left(\mu\frac{b_1}{c_1}\right)+(c_2)^2\coth\left(\mu\frac{b_2}{c_2}\right)\right]. \tag{121}$$

We compute further

$$\mu_h = e(c_1-c_2)\left[\frac{\gamma_1\gamma_2}{\gamma_1 c_1+\gamma_2 c_2}+\frac{(c_1)^2+c_1c_2+(c_2)^2}{R}\right] \tag{122}$$

$$E_h^{1212} = \mu_h A_h^{1212}, \tag{123}$$

$$P_h/C^{1212} = \frac{2}{3}\left[\gamma_1(c_1)^3+\gamma_2(c_2)^3-\frac{1}{R}[(c_1)^3-(c_2)^3]^2\right], \tag{124}$$

$$D_h^{1212} = P_h+(\mu_h)^2A_h^{1212}. \tag{125}$$

5.3. *Stiffnesses of indices* $(\alpha\alpha\beta\beta)$

On making use of the results of Sections 4.5 and 5.1, one arrives at the formulae for the stiffnesses $A_h^{\alpha\alpha\beta\beta}$:

$$A_h^{1111}/\tilde{C}^{1111} = \frac{1}{S}[\gamma_1(c_2)^3+\gamma_2(c_1)^3], \tag{126}$$

$$S = \frac{1}{2}\left[\left(\frac{\gamma_1}{c_1}+\frac{\gamma_2}{c_2}\right)[\gamma_1(c_2)^3+\gamma_2(c_1)^3]+3\gamma_1\gamma_2e^2\right] \tag{127}$$

$$A_h^{1122} = A_h^{2211} = \nu_{12}A_h^{1111}, \quad \nu_{12} = \tilde{C}^{1122}/\tilde{C}^{1111} \tag{128}$$

$$A_h^{2222} = 2\bar{E}(\gamma_1c_1+\gamma_2c_2)+(\nu_{12})^2A_h^{1111}, \tag{129}$$

where

$$\bar{E} = \tilde{C}^{2222}-(\tilde{C}^{1122})^2/\tilde{C}^{1111} \text{ or } \bar{E} = (1-\nu_{12}\nu_{21})\tilde{C}^{2222}, \quad \nu_{21} = \tilde{C}^{1122}/\tilde{C}^{2222}. \tag{130}$$

The reciprocal stiffnesses read

$$E_h^{1111} = \frac{\gamma_1\gamma_2[(c_1)^4-(c_2)^4]e}{(\gamma_1c_1+\gamma_2c_2)S}\tilde{C}^{1111} \tag{131}$$

$$E_h^{1122} = E_h^{2211} = \nu_{12}E_h^{1111}, \quad E_h^{2222} = (\nu_{12})^2E_h^{1111}. \tag{132}$$

Moreover, $F_h^{\alpha\alpha\beta\beta} = E_h^{\beta\beta\alpha\alpha}$. The bending stiffnesses are expressed as follows:

$$\begin{aligned} D_h^{1111} &= \left[\frac{(c_1c_2)^3}{3S}\left(\frac{\gamma_1}{c_1}+\frac{\gamma_2}{c_2}\right)+\frac{e^2}{S}\frac{\gamma_1\gamma_2[\gamma_1(c_1)^5+\gamma_2(c_2)^5]}{(\gamma_1c_1+\gamma_2c_2)^2}\right]\tilde{C}^{1111}, \\ D_h^{1122} &= D_h^{2211} = \nu_{12}D_h^{1111}, \\ D_h^{2222} &= 2\bar{E}\left[\frac{\gamma_1(c_1)^3+\gamma_2(c_2)^3}{3}+\frac{\gamma_1\gamma_2e^2c_1c_2}{\gamma_1c_1+\gamma_2c_2}\right]+(\nu_{12})^2D_h^{1111}. \end{aligned} \tag{133a–c}$$

One can check that the expression

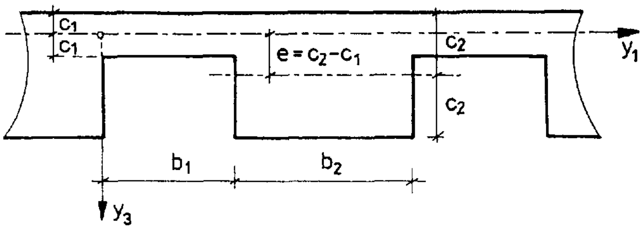

Fig. 2. Case of a flat upper face.

$$J_c = \tfrac{2}{3}a[\gamma_1(c_1)^3+\gamma_2(c_2)^3]+2ae^2\frac{\gamma_1\gamma_2c_1c_2}{\gamma_1c_1+\gamma_2c_2} \tag{134}$$

represents the moment of inertia of the cross-section $\mathscr{Y}$ with respect to its own neutral axis (cf. Fig. 1). Thus, eqn (133c) can be rearranged into the form

$$D_h^{2222} = \frac{\bar{E}J_c}{a}+(\nu_{12})^2D_h^{1111}. \tag{135}$$

Note that if we replace

$$c_1 \to c_2,\quad c_2 \to c_1,\quad b_1 \to b_2,\quad b_2 \to b_1,\quad e \to -e,$$

then expressions (126)–(133) do not change their form. This means that the formulae do not depend upon the choice of the periodicity cell.

Consider an isotropic plate with stepwise varying lower face (see Fig. 2). The geometry of the periodicity cell is determined by b_α, c_α, $a = b_1+b_2$. The eccentricity e equals c_2-c_1. The formulae found above apply well to the case considered.

5.4. *Effective stiffnesses in the case of transverse symmetry*

The $\mathscr{Y}$ cell is transversely symmetric if $e = 0$. We put e equal to zero in eqns (126)–(133). Stiffnesses $E_h^{\alpha\beta\lambda\mu}$, $F_h^{\alpha\beta\lambda\mu}$ vanish. The non-zero stiffnesses read

$$A_h^{1111} = \left(\frac{\gamma_1}{2c_1}+\frac{\gamma_2}{2c_2}\right)^{-1}\tilde{C}^{1111},\quad A_h^{1122} = \nu_{12}A_h^{1111}, \tag{136}$$

$$A_h^{2222} = 2(\gamma_1c_1+\gamma_2c_2)\bar{E}+(\nu_{12})^2A_h^{1111},\quad A_h^{1212} = \left(\frac{\gamma_1}{2c_1}+\frac{\gamma_2}{2c_2}\right)^{-1}C^{1212}$$

and

$$D_h^{1111} = \frac{2}{3}\left[\frac{\gamma_1}{(c_1)^3}+\frac{\gamma_2}{(c_2)^3}\right]^{-1}\tilde{C}^{1111},\quad D_h^{1122} = \nu_{12}D_h^{1111}$$

$$D_h^{2222} = \tfrac{2}{3}[\gamma_1(c_1)^3+\gamma_2(c_2)^3]\bar{E}+(\nu_{12})^2D_h^{1111},$$

$$D_h^{1212} = \tfrac{2}{3}[\gamma_1(c_1)^3+\gamma_2(c_2)^3-\frac{1}{R}[(c_1)^3-(c_2)^3]^2]C^{1212}, \tag{137}$$

where R is given by eqn (121).

Note that in the case considered ($e = 0$), only the torsional stiffness is sensitive to the ratios $b_\alpha : c_\alpha$ determining the transverse shape of the periodicity cell.

In the case of isotropy, one should assume

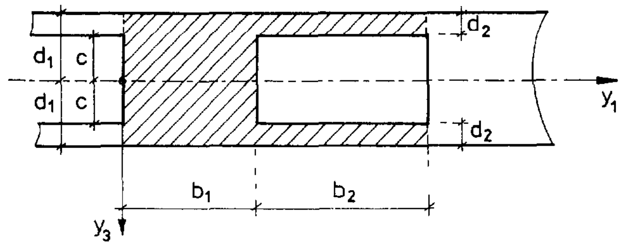

Fig. 3. Hollow plate. The dashed area represents the periodicity cell $\mathscr{Y}$.

$$\tilde{C}^{1111} = \tilde{C}^{2222} = \frac{E}{1-\nu^2}, \quad \nu_{12} = \nu, \quad C^{2323} = C^{1212} = G_s = \frac{E}{2(1+\nu)}, \quad \bar{E} = E, \tag{138}$$

where E stands for the Young's modulus and ν is the Poisson ratio.

6. EFFECTIVE STIFFNESSES OF A HOLLOW PLATE

Consider a transversely symmetric plate, made of an orthotropic material with orthotropy axes y_i, having rectangular openings located symmetrically (see Fig. 3). Stiffnesses $A_\sigma^{\alpha\beta\lambda\mu}, D_\sigma^{\alpha\beta\lambda\mu}$ are computed according to eqns (21). We find

$$A_\sigma^{\alpha\beta\lambda\mu} = 2d_\sigma \tilde{C}^{\alpha\beta\lambda\mu}, \quad H_\sigma^{\alpha\beta} = 2\kappa d_\sigma C^{\alpha 3\beta 3},$$
$$D_1^{\alpha\beta\lambda\mu} = \tfrac{2}{3}(d_1)^3 \tilde{C}^{\alpha\beta\lambda\mu}, \quad D_2^{\alpha\beta\lambda\mu} = \tfrac{2}{3}[(d_1)^3 - (d_1-d_2)^3]\tilde{C}^{\alpha\beta\lambda\mu} \tag{139}$$

6.1. *Stiffnesses of indices* (1212)

By using results found in Section 4.5, one obtains

$$A_h^{1212} = \left(\frac{\gamma_1}{2d_1} + \frac{\gamma_2}{2d_2}\right)^{-1} C^{1212},$$

$$D_h^{1212} = \frac{2}{3}\left[\gamma_1(d_1)^3 + \gamma_2(1-3\zeta+3\zeta^2)(d_2)^3 - \frac{(d_1-d_2)^6}{T}\right]C^{1212}, \tag{140}$$

where

$$T = a\left[\mu(d_1)^2 \coth\left(\frac{\mu b_1}{d_1}\right) + \mu'(d_2)^2 \coth\left(\frac{\mu'' b_2}{d_2}\right)\right], \quad \zeta = d_1/d_2$$

$$\mu' = (1-3\zeta+3\zeta^2)^{1/2}\mu, \quad \mu'' = (1-3\zeta+3\zeta^2)^{-1/2}\mu. \tag{141}$$

6.2. *Stiffnesses of indices* ($\alpha\alpha\beta\beta$)

On using results found in Section 4.5, one arrives at

$$A_h^{1111} = \left(\frac{\gamma_1}{2d_1} + \frac{\gamma_2}{2d_2}\right)^{-1} \tilde{C}^{1111}, \quad A_h^{1122} = \nu_{12} A_h^{1111},$$

$$A_h^{2222} = 2(\gamma_1 d_1 + \gamma_2 d_2)\bar{E} + (\nu_{12})^2 A_h^{1111};$$

$$D_h^{1111} = \frac{2}{3}\left[\frac{\gamma_1}{(d_1)^3} + \frac{\gamma_2}{(d_1)^3 - (d_1-d_2)^3}\right]^{-1} \tilde{C}^{1111}, \quad D_h^{1122} = \nu_{12} D_h^{1111},$$

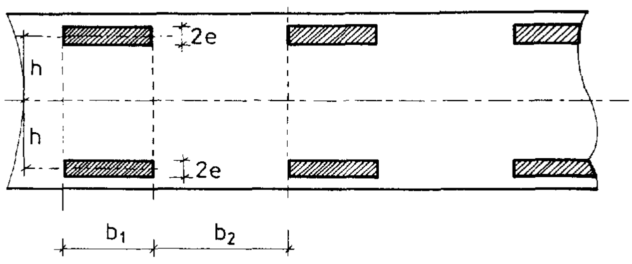

Fig. 4. Plate with reinforcement located transversely symmetric.

$$D_h^{2222} = \tfrac{2}{3}\bar{E}[(d_1)^3 - \gamma_2(d_1 - d_2)^3] + (\nu_{12})^2 D_h^{1111}, \tag{142}$$

where $\bar{E}$ has been defined by eqn (130).

7. EFFECTIVE TORSIONAL STIFFNESS OF A REINFORCED PLATE

Consider a plate made of isotropic elastic material of moduli C_m^{ijkl}, strengthened transversely symmetric by a reinforcement made of isotropic material of moduli C_f^{ijkl} (cf. Fig. 4). Let

$$\eta = C_f^{1212}/C_m^{1212}, \quad \varepsilon_1 = e/c, \quad \varepsilon_2 = h/c, \quad \varepsilon_3 = a/c. \tag{143}$$

According to eqn (106) the effective torsional stiffness D_h^{1212} of the plate considered is given by

$$D_h^{1212}/D_m^{1212} = \gamma_1\delta_1 + \gamma_2 - \frac{(\delta_1 - 1)^2}{\delta_2}, \quad D_m^{1212} = \tfrac{2}{3}c^3 C_m^{1212}, \tag{144}$$

where

$$\delta_1 = 1 + 2(\eta - 1)[(\varepsilon_1)^3 + 3\varepsilon_1(\varepsilon_2)^2], \quad \delta_2 = \tfrac{1}{2}\chi_1\delta_1 \coth(\tfrac{1}{2}\chi_1\gamma_1) + \tfrac{1}{2}\chi_2 \coth(\tfrac{1}{2}\chi_2\gamma_2), \tag{145}$$

$$\chi_2 = \sqrt{3\kappa\varepsilon_3}, \quad \chi_1 = \left(\frac{1 + 2(\eta - 1)\varepsilon_1}{\delta_1}\right)^{1/2}\chi_2. \tag{146}$$

The formulae for the remaining stiffnesses can be found similarly, by applying the formulae of Section 4.5.

8. ANALYSIS OF FORMULAE FOR EFFECTIVE STIFFNESSES

8.1. *Influence of slenderness of the periodicity cell*

Consider the plate of Fig. 2. Let us fix $\gamma_\alpha = b_\alpha/a$, c_1, c_2. We examine a family of isotropic plates indexed by the slenderness parameter $\lambda = a/c_2$. Note that stiffnesses of indices $(\alpha\alpha\beta\beta)$ do not depend upon λ; only stiffnesses of indices (1212) are sensitive to the slenderness of the $\mathscr{Y}$ cell. Together with increase of λ the stiffnesses $A_h^{1212}, E_h^{1212}, D_h^{1212}$ grow to certain

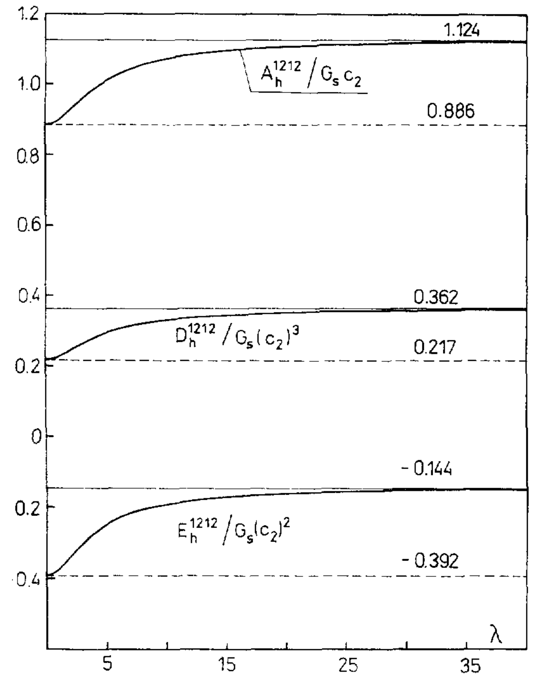

Fig. 5. Case of Fig. 2. Dimensions c_1 and c_2 are fixed. Influence of the $\lambda = a/c_2$ parameter.

asymptotic values (see Fig. 5) (case: $\sigma = 2/5$, $\gamma_1 = 1/2$, $\kappa = 5/6$). In Fig. 5 these stiffnesses are normalized with respect to $G_s(c_2)^i$, $G_s = 0.5E/(1+\nu)$, $i = 1,2,3$, respectively.

8.2. *The same plate: influence of the ratio* $\sigma = c_1/c_2$

Consider the family of isotropic plates of Fig. 2 such that vol($\mathscr{Y}$) = const, vol($\mathscr{Y}$) = $2Va^2 = 2(b_1c_1+b_2c_2)$. The quantity $\sigma = c_1/c_2$ plays the role of a variable. It is sufficient to consider the interval $\sigma \in [0,1]$. We fix V, γ_1, κ, ν and then determine

$$c_2(\sigma) = \frac{Va}{\gamma_1\sigma+\gamma_2}, \quad c_1(\sigma) = \sigma c_2(\sigma). \tag{147}$$

The effective stiffnesses are hence computed at fixed V, γ_1, κ and ν for subsequent values of σ. The results for the data $V = 1.0$, $\gamma_1 = 1/2$, $\kappa = 5/6$ and $\nu = 0.3$ are given in Figs 6 and 7. The remaining stiffnesses are computed according to

$$A_h^{1122} = \nu A_h^{1111}, \quad D_h^{1122} = \nu D_h^{1111}, \quad E_h^{1122} = \nu E_h^{1111}, \quad E_h^{2222} = \nu^2 E_h^{1111}. \tag{148}$$

At the point $\sigma = 0$, referring to the case when the plate decomposes into independent beams, only stiffnesses $D_h^{1212}, A_h^{2222}, D_h^{2222}$ do not vanish. The graphs of $P_h = D_h^{1212}(e = 0)$ [cf. eqn (124)] refer to the plate with a plane of symmetry. Note that for $\sigma = 0$ and $\sigma = 1.0$, stiffnesses P_h and D_h^{1212} assume the same values.

The coupling constants $E_h^{\alpha\beta\lambda\mu}$ vanish at both ends: $\sigma = 0$, $\sigma = 1$. Within the interval $[0,1]$ they assume one minimum of negative value.

8.3. *Reinforced plate*

To analyse the sensitivity of the effective torsional stiffness of the plate of Fig. 4 with respect to the ratio $\varepsilon_1 = e/c$, $0 < \varepsilon_1 \leqslant 0.5$, representing the relative amount of reinforcement, and with respect to $\eta = C_f^{1212}/C_m^{1212}$, we fix ε_2, ε_3, γ_1 and κ. To be specific we put $\varepsilon_2 = 0.5$, $\varepsilon_3 = 5.0$, $\gamma_1 = 0.5$, $\kappa = 5/6$ and consider four values of ε_1: 0.0, 0.1, 0.3 and 0.5 (see Fig. 8). The curves $D_h^{1212}(\eta)/D_m^{1212}$ cross at point (1,1), referring to a homogeneous case. If ε_1 tends to zero, the curves considered tend to the constant function $\equiv 1$.

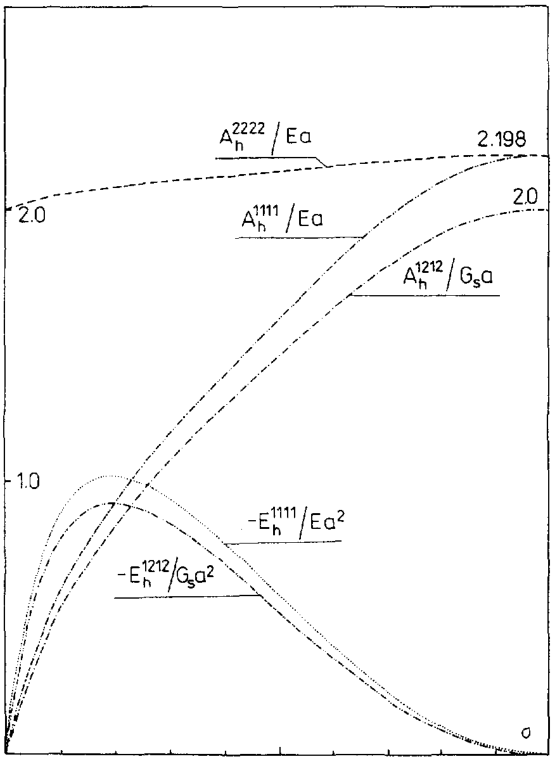

Fig. 6. Case of Fig. 2 with $\gamma_1 = 1/2$, $\mu = 0.3$. Plates of constant volume $2a^2$. Stiffnesses $A_h^{\alpha\beta\lambda\mu}$, $E_h^{\alpha\beta\lambda\mu}$ versus $\sigma = c_1/c_2$.

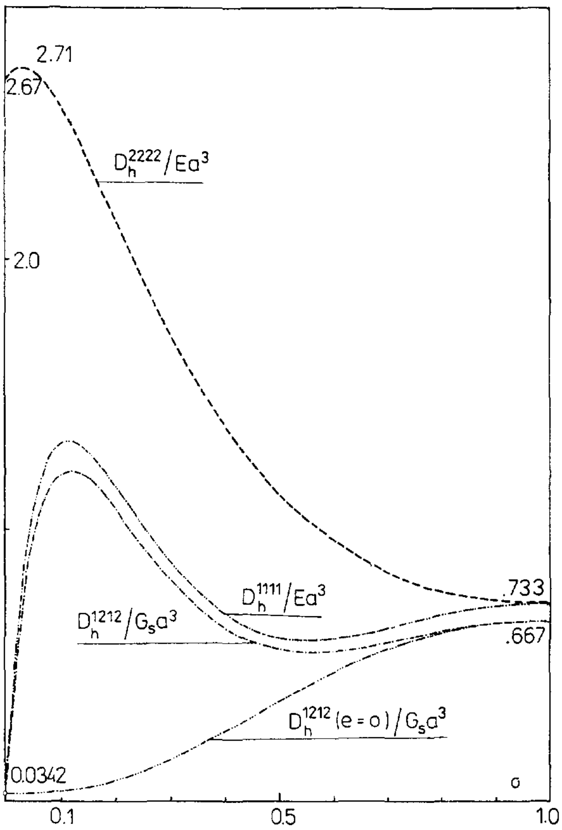

Fig. 7. The same family of plates. Stiffnesses $D_h^{\alpha\beta\lambda\mu}$ versus $\sigma = c_1/c_2$.

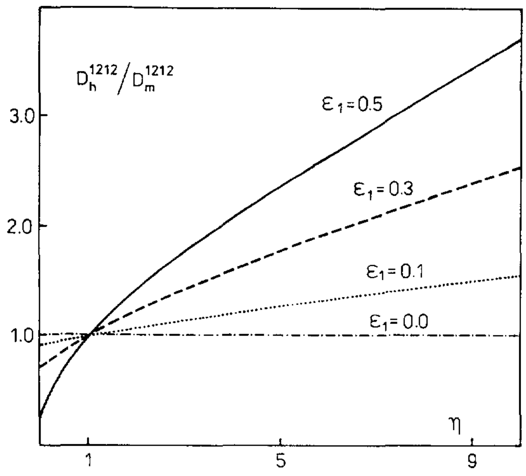

Fig. 8. Reinforced plates of Fig. 4. Case of $\varepsilon_2 = 0.5$, $\varepsilon_3 = 5$, $\gamma_1 = 0.5$. Torsional stiffness versus η.

9. FINAL REMARKS

The method presented is approximate if compared with the Caillerie (1984, $e \approx \varepsilon$) and Kohn–Vogelius (1984, 1985, $a = 1$) perfect methods of averaging and can be viewed as refined with respect to the hitherto existing averaging methods based upon the conventional scalings, as proposed in Duvaut and Metellus (1976) for thin balanced (transversely symmetric) plates and in Bourgeat and Tapiéro (1983) for moderately thick balanced plates. Its relation with respect to the aforementioned averaging methods follows clearly from Fig. 5. The upper asymptote for X_h^{1212}, $X = A, E, D$, refers to the in-plane (conventional) scaling of Hencky-type problems ($\hat{P}^{\alpha}_{\mathrm{loc}}$) (cf. Section 3.1). In the transversely symmetric case, this method was originated by Bourgeat and Tapiéro (1983). Thus, the upper asymptotes can be viewed as Bourgeat–Tapiéro type approximations. On the other hand, the lower asymptotes can be interpreted as results based upon the $e \to 0$, then $\varepsilon \to 0$ model of Caillerie (1984) and the $a < 1$ model of Kohn and Vogelius (1984, 1985). These results can also be viewed as Duvaut-type approximations, since they can be arrived at by the asymptotic homogenization of the relevant Kirchhoff problem for the asymmetric plates considered, if based on conventional scaling, as used in Duvaut and Metellus (1976). Thus, the formulae for effective stiffnesses reported here provide us with results lying between predictions of the two two-dimensional averaging methods based upon in-plane scalings.

The formulae reported are ready for applications in the regularized optimization problems of anisotropic transversely asymmetric plates.

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